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NOTE ON DISPERSION FORMULAE AND THE SECONDARY SPECTRUM

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Abstract. The basis from which dispersion formulae should be derived for optical glass is considered, and expressions depending on a series of powers of the wave length investigated, using the figures given in the various Jena glass lists as the criterion for the selection of the index. To obtain further information of the accuracy of the resulting formula, it is applied to Gifford's figures. The comparison in general confirms the conclusion that the expression found, which only differs slightly from Conrady's second formula, is satisfactory, and further leads to the view that Gifford's results may require correction by amounts tabulated. The bearing of the formula on the secondary spectrum of thin lenses is discussed.

THE importance of securing freedom from the secondary spectrum in lens systems of long focus has long been recognised, and many investigations relating to this object have been published. Among these a number relate to expressions which indicate to varying degrees of accuracy the changes of refractive index of optical glasses with the wave length. Recent papers give the impression that formulae which were proposed some years back, such as those of Conrady, are not appreciated at their real value, for other expressions of lower accuracy and less convenient form have been suggested as improvements on those already in use. It may therefore not be superfluous to reconsider the problem and to present the evidence for or against a particular expression in a way which will permit a definite conclusion to be drawn without hesitation.

There are two obvious methods which may be utilised in deriving dispersion formulae. One is to study the observations of index for many wave lengths on a small number of glasses, and the other is to base the investigation on the index

values for a few wave lengths of a large number of glasses. The observations of Gifford serve as a suitable foundation for the former method, and the values given in the glass catalogues of manufacturers of optical glass, such as Chance Bros., The Derby Crown Glass Co., Parra-Mantois et Cie., and Schott u. Gen., for the latter method. For various reasons, among which may be mentioned the importance of securing a formula which will as far as possible be applicable to *all* glasses on substituting suitable values for the general constants, the latter method, in which all varieties of glass are taken into account, is adopted here as a basis, and it will subsequently be seen how closely it accords with observations such as those of Gifford.

An examination of manufacturers' lists shows that it cannot be claimed that any maker has entirely succeeded in avoiding obvious errors in his published values, and in some cases these are serious. Internal evidence shows that some are particularly prone to mistakes. Among the most satisfactory are those of the Jena glass works, and it will be convenient to adopt these as a basis both for their general accuracy and on account of the extensive range of glasses which has been produced since they opened in the middle eighties of last century. Many of the extreme glasses included in the earlier lists of this firm have since been deleted as they proved deficient in stability or suffered from other serious defects, but for a purpose like that now in hand it is important that such glasses should not be left out of account. The Jena lists give the refractive index for the mean sodium lines, indicated by the suffix D, to ·0001, and the differences of index between this line and the potassium line A' and the hydrogen lines C, F, G' to ·00001. The individual measurements of the differences in these intervals are said to be accurate to ± ·00002. If it is supposed that an error of + ·00002 is made in the value of $\mu_F - \mu_C$ and an error of − ·00002 in the value of $\mu_{G'} - \mu_F$, the two corresponding to a slight error in the setting upon F, a discrepancy between the observations and the values derived from a formula will result, and it will later be evident that this discrepancy will amount very approximately in the case of the last interval to ·00005. Discrepancies within this amount are not therefore necessarily significant. Detailed calculation, however, shows that the instances in which the divergence between observation and calculation reaches such an amount are few, and among the glasses sufficiently stable and colourless to be of general utility, distinctly rare. In a proportion of these rare instances the measurements of Gifford result in more normal agreement between observed and calculated values.

If the values of the change in refractive index for two of the intervals mentioned above are expressed as proportions of a third interval, and the positions of various glasses identified by these two ratios as variables are plotted on a chart using rectangular axes, as in Fig. 1, the points representing the glasses will be found to lie very closely on a straight line. This is true whichever intervals are taken. Let the best mean line through this aggregate of points be chosen and its Cartesian equation found. Glasses which have their representative points on this line may be considered 'normal' in their dispersive properties, and in the absence of more accurate measurements glasses which depart from normal properties by amounts which may

be due to errors of observation may also be regarded as normal. It follows that for all but a very few exceptional glasses the correct formula will involve relations represented by the straight line of this diagram.

Distinguish for the moment chosen wave lengths for which measurements have been made by numerical suffixes. The equation to the straight line may be written in the form $$P(\mu_1 - \mu_2) + Q(\mu_3 - \mu_4) + R(\mu_5 - \mu_6) = 0 \quad \ldots\ldots(1),$$ where the ratios $P : Q : R$ are determined by the observations. The general expression for the refractive index will take some such form as

$$\mu - \mu_0 = a\{f(\lambda) - f(\lambda_0)\} + b\{g(\lambda) - g(\lambda_0)\} + c\{h(\lambda) - h(\lambda_0)\} + \ldots \ldots(2),$$

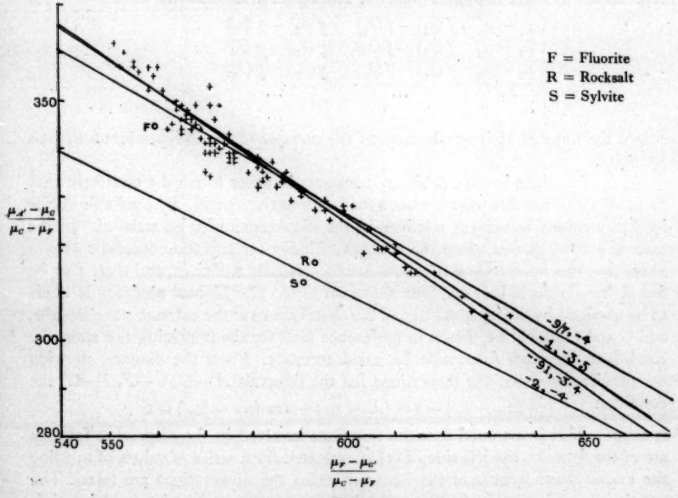

Fig. 1.

where λ denotes the wave length for which μ is the index, a, b, c, ... are general constants which have definite values for any one glass and are independent of the wave length, and f, g, h, ... are functions of the wave length which will not change for normal glasses. The substitution of this expression for the index in the condition defining normal glasses gives

$$0 = \Sigma a\,[P\{f(\lambda_1) - f(\lambda_2)\} + Q\{f(\lambda_3) - f(\lambda_4)\} + R\{f(\lambda_5) - f(\lambda_6)\}].$$

In so far then as the general expression sought contains the constants typical of the glasses in the form given in (2), the various functions of λ must satisfy conditions of the form

$$P\{f(\lambda_1) - f(\lambda_2)\} + Q\{f(\lambda_3) - f(\lambda_4)\} + R\{f(\lambda_5) - f(\lambda_6)\} = 0 \ldots\ldots(3).$$

The number of functions necessary depends on the number of degrees of freedom in the glasses apart from the dispersive relations. Evidently at least two functions are required in the present case since there is a continuous change in the relative partial dispersions on proceeding from light to dense glasses. If the class of function is assigned, (3) is a condition enabling the proper members of the class to be chosen.

The converse of this result also holds. If the functions in (2) are given, linear relations corresponding to (1) can be found, and thus a line can be drawn on a suitable diagram on which the points representing glasses whose dispersions are represented by these functions must lie. The equation is evidently

$$0 = \begin{vmatrix} \mu_1 - \mu_2 & f(\lambda_1) - f(\lambda_2) & g(\lambda_1) - g(\lambda_2) & \ldots\ldots \\ \mu_3 - \mu_4 & f(\lambda_3) - f(\lambda_4) & g(\lambda_3) - g(\lambda_4) & \ldots\ldots \\ \mu_5 - \mu_6 & f(\lambda_5) - f(\lambda_6) & g(\lambda_5) - g(\lambda_6) & \ldots\ldots \\ \ldots & \ldots & \ldots & \ldots\ldots \\ \ldots & \ldots & \ldots & \ldots\ldots \end{vmatrix}$$

where the number of intervals exceeds the number of the independent functions by unity.

If the functions involve arbitrary constants in other forms the treatment may be modified in suitable ways to obtain the relations they imply. By a suitable choice in the functions non-linear relations in the dispersions may be satisfied. In the case of normal glasses, since the changes of index for any three intervals obey a linear law, two independent functions are theoretically sufficient, and these may be found from observations for four spectrum lines. The highest accuracy is likely to be obtained by taking advantage of the observations at the extreme wave lengths, and C and F should be chosen in preference to D for the remaining two since the conditions are more favourable for good accuracy. From the diagram showing the relations between the dispersions for the intervals A'—C, C—F, F—G' the condition

$$178\,(\mu_{A'} - \mu_C) - 131\,(\mu_C - \mu_F) + 122\,(\mu_F - \mu_{G'}) = 0$$

is found. If it is assumed that the functions involved in the dispersion formula are of the form λ^n, the left side of (3) is evaluated for a series of values of n, using the known wave lengths of the lines for which the observations are taken. The solution $n = 0$ is meaningless, and Fig. 2 shows that the only values which satisfy the condition are approximately $n = -0{\cdot}91$ and $n = -3{\cdot}4$. These values will be confirmed if a different combination of four lines is selected, using the figures from the Schott catalogues, and therefore it may be assumed that these values of n represent the dispersion of glass as closely as is possible with simple power functions of the wave lengths, assuming that the figures of these catalogues form a reliable basis.

It will be of interest to note what relations between these dispersions are involved if other expressions which have been proposed are adopted. A few of these are included in Fig. 1. The two values of n involved are recorded close to the line to which they relate. Of these the one most frequently used is the Cauchy form

$$\mu = \mu_0 + a\lambda^{-2} + b\lambda^{-4},$$

though this relation is evidently far from being satisfied by any optical glass. The

two forms suggested by Conrady are (1) $n = -9/7$ and -4, and (2) $n = -1$ and $-7/2$, and the latter is a very distinct improvement on all previous expressions. No improvement is to be expected on this when very simple forms for n are required, and the differences between the results which it gives and those derived from the expression which best fits the observations recorded in the Jena lists are small.

Since only two functions are involved for the complete determination of the properties of normal glasses, only one degree of freedom remains when one dispersion is known. More generally if one dispersion is known a relation is found

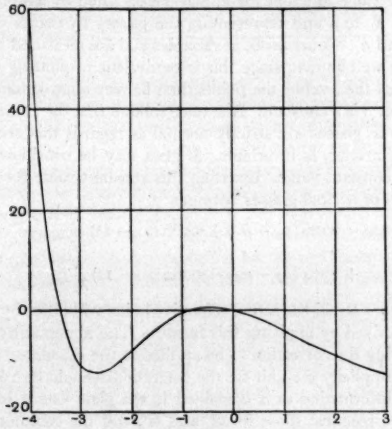

Fig. 2.

between the unknown glass constants $a, b, c, \ldots$ of (2). If this is rewritten in the form

$$\mu - \mu_0 = \Sigma a' \frac{f(\lambda) - f(\lambda_0)}{f(\lambda_3) - f(\lambda_4)},$$

where the dispersion is known for the interval λ_3 to λ_4, this relation is

$$\mu_3 - \mu_4 = \Sigma a'.$$

Assuming the interval C to F for the known measurements, the dispersive expression is reducible to the form

$$\mu - \mu_D = \{\alpha(\mu_F - \mu_C) + \beta(\mu_D - 1)\} \frac{f(\lambda) - f(\lambda_D)}{f(\lambda_F) - f(\lambda_C)} + \{(1 - \alpha)(\mu_F - \mu_C) - \beta(\mu_D - 1)\} \frac{g(\lambda) - g(\lambda_D)}{g(\lambda_F) - g(\lambda_C)} \quad \ldots\ldots(4),$$

where the facts so far considered impose no restrictions on the values of α and β. In effect the combination of the two constitutes only a single variable, and α may

therefore be given a constant value chosen to make β as small as possible for glasses in general. If then α has the same value for all glasses, in any combination of thin lenses corrected for chromatic aberration for C and F

$$\kappa - \kappa_D = \left\{ \begin{matrix} f(\lambda) - f(\lambda_D) \\ f(\lambda_F) - f(\lambda_C) \end{matrix} - \begin{matrix} g(\lambda) - g(\lambda_D) \\ g(\lambda_F) - g(\lambda_C) \end{matrix} \right\} \Sigma \kappa_D \beta,$$

so that the character of the secondary spectrum in thin combinations of normal glasses is invariable, and its amount depends on the value $\Sigma \kappa_D \beta$.

Equation (4) shows that the value of β for any given variety of glass, as well as the most suitable value to adopt for α, may be obtained by assigning a particular value other than λ_D to λ and representing the glasses by points on a diagram in which $\mu_F - \mu_C$ and $\mu_D - 1$ are taken as variables and are measured in perpendicular directions. As is well known, since this is equivalent to plotting the ratio of two dispersions against the ν value, the points then lie very approximately on a straight line, implying that β is a constant. It at once follows that the amount of secondary spectrum when the glasses are strictly normal as regards this second linear law, like its general character, is invariable. A glass may be considered quite normal when β has this constant value. Inserting this constant value for β, the equation for the dispersion of normal glasses becomes

$$\mu - \mu_D = \{\cdot 226\,(\mu_F - \mu_C) + \cdot 0062\,(\mu_D - 1)\} \frac{\lambda^{-\cdot 91} - \lambda_D^{-\cdot 91}}{\lambda_F^{-\cdot 91} - \lambda_C^{-\cdot 91}}$$
$$+ \{\cdot 774\,(\mu_F - \mu_C) - \cdot 0062\,(\mu_D - 1)\} \frac{\lambda^{-3\cdot 4} - \lambda_D^{-3\cdot 4}}{\lambda_F^{-3\cdot 4} - \lambda_C^{-3\cdot 4}} \qquad \ldots\ldots(5).$$

When a glass is not normal the actual refractive index will differ by a small amount from the value derived by applying this formula. The abnormality is most clearly shown by tabulating the correction to be applied to the calculated value to get the true value. For simplicity the unit for the corrections should be 10^{-5}. To give the same amount of information as is contained in the glass lists it is only necessary to record three corrections, those for A' and G', and the common correction for C and F. The latter correction can actually be dispensed with since it hardly ever exceeds 2 numerically, and this is the figure representing the accuracy of the observation. The actual number of cases where this correction exceeds 2 among the glasses that have not been withdrawn is three with indices less than 1·7, the numerical value of the correction being 3 in each case, and three glasses with index above 1·7, two having the value 3 and one the value 5.

Of greater significance are the corrections for the extreme red and extreme visible violet. Out of a total of 132 glasses the corrections are distributed as follows:

	Red	Violet
Beyond − 10	9	10
− 10 or − 9	4	1
− 8 or − 7	3	3
− 6 or − 5	9	4
− 4 or − 3	8	7
− 2 or + 2	61	81
+ 3 or + 4	21	10
+ 5 or + 6	6	6
+ 7 or + 8	2	4
+ 9 or + 10	3	0
Beyond + 10	6	6

If ± 2 represents a likely variation for normal glasses, practically all results would be expected to lie between ± 6. Of the glasses that have not been withdrawn those that lie outside these limits in the red are one telescope flint with the value -8 and at the other end four extreme barium glasses with the values 7, 9, 10, 11 and three very dense flints with the index exceeding 1·7 all of which have the value 11. It may, however, be remarked in passing, for what it is worth, that Gifford has given values for three of these glasses containing barium, and obtains indices which correspond to the corrections 3, 3 and 5. In any case the evidence is extremely strong that ± 6 covers the glasses of importance.

Greater importance attaches to the violet end of the spectrum and the results here are even more remarkable. Apart from glasses which have an index in excess of 1·7, the only varieties still listed which have corrections exceeding ± 6 are two telescope flints with -7 and two dense flints with 7 and 8 respectively. The situation may therefore almost be summed up by saying that more refined measurements are necessary to demonstrate that stable glasses depart from the values given by (5) in the visible spectrum. The chief evidence to the contrary is derived from the observation that particular groups of glasses—for instance the densest barium crowns—on the whole exhibit a tendency to depart from this law in a definite direction.

It is important that the results obtained in this way should be checked against observations on several lines for a number of glasses. For this purpose it will hardly be disputed that the results published by Gifford merit first consideration both on account of the number of lines for which results are given and of the care exercised in the measurements and in deducing the results. The comparison is interesting from another point of view, for it may well prove that it leads to a criterion for determining the accuracy attained in the various observations in addition to the verification of the suggested law. If the law holds satisfactorily the divergencies between calculation and observation will be comparable throughout the range, with perhaps greater unsystematic differences at the extremes due to the greater difficulty experienced in setting. The law will only be shown to be inapplicable if regular and systematic departures are found.

It must be assumed in making the comparison that errors may be present in any particular measurements, and in particular an error may be made in the measurement of $\mu_F - \mu_C$, and in consequence all the calculated results will be affected. The comparison should allow of the correction of any such error which there is reason to believe present. The most rapid way of exhibiting the results is to plot the corrections for each wave length on the assumption that μ_F and μ_C are accurate, and choose the scale in the perpendicular direction so that with normal glasses an error in this particular interval would be corrected by a tilting of the zero line. The simplest smooth curve should then be drawn in a mean position through the points. In the absence of evidence to the contrary it is reasonable to suppose that the corrections to the calculated values may be read from this curve with greater accuracy than from the individual points, the departures of the points from the curve representing probable errors in the individual observations. The test on the observations is found to be very severe, and though

it can hardly be laid down that the discrepancies between the points and the smooth curve are necessarily errors of observation, the presumption that this is correct, particularly in view of the general accuracy with which this formula represents the properties of optical glasses generally, is sufficiently strong to justify the adoption of the curve rather than the observed points until independent evidence shows that the apparent irregularities are real.

A few examples are given of the comparison obtained between the calculated and observed results. The figures used are those given by Lamplough and Mathews in the *Transactions of the Optical Society*, Vol. 19 (1918), p. 83. They are Gifford's observations corrected for barometric pressure. The mean dispersion is given to

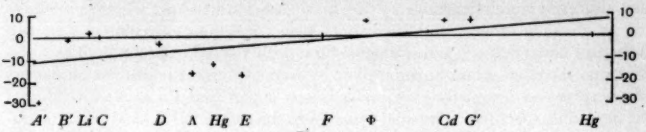

Fig. 3.

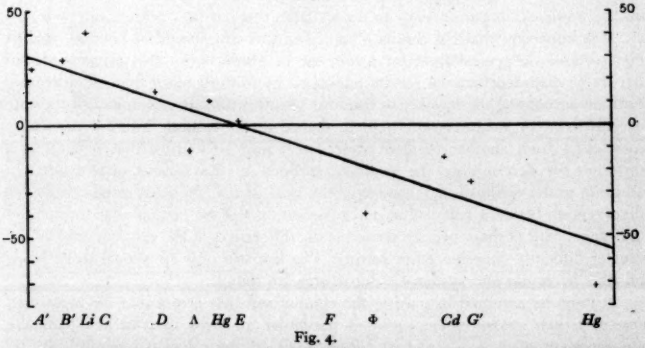

Fig. 4.

five decimal places, but four significant figures are given for the partial ratios and the figures can therefore be considered as though the sixth decimal place were given.

Fig. 3 shows the corrections found for glass No. 3, the unit for the vertical scale being unity in the sixth decimal place in the index. The best curve through these points is the oblique straight line drawn, and the glass is considered normal. The observations agree with the curve within about one-fifth place unit except in the case of *E* where the difference is 1·3 and *A′* where it amounts to 1·8. An example showing greater errors for a normal glass is No. 6 illustrated in Fig. 4. The errors for *C* and *F* appear to be of opposite signs, but when a correction is applied for this interval the greatest discrepancy becomes 2 in the fifth place instead of 7 as given by the original comparison.

In the case of glasses which are not strictly normal the best curve is roughly parabolic. An example occurs in Fig. 5 illustrating glass No. 8. The intersections of the curve with the ordinates on which the readings for *C* and *F* are plotted

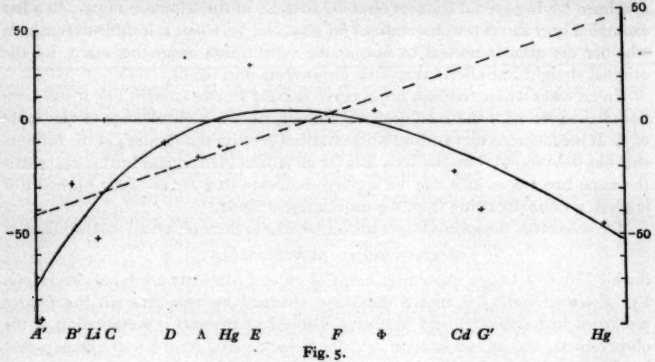

Fig. 5.

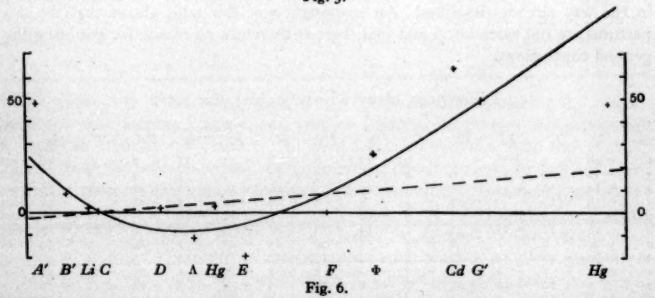

Fig. 6.

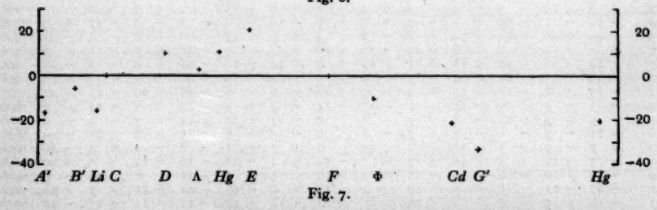

Fig. 7.

determine the corrected reference line shown dotted, and the probable corrections to the formula are given by the vertical separation between the curve and this straight line. The shape of this curve is characteristic of telescope flints, many of

which exhibit more pronounced departures from the normal. In this instance the most serious difference between a point and the curve is at C, and amounts to three fifth place units. The next example, Fig. 6, illustrates glass No. 9, the only specimen having special features opposed to those of the telescope flints. As a last example Fig. 7 shows the observations for glass No. 36, where it is difficult to decide whether the glass is normal or not, as the substitution of another curve for the original straight line effects very little improvement in the fit.

In all cases where the best mean curve is not a simple straight line it has been found to agree with the modification resulting from a slight change in the value of β. It is concluded that Gifford's observations confirm the accuracy of the formula that has been found from the Jena lists for all optical glasses, and further show that the same two values of n may be applied to glasses that are not fully normal if β is given a value departing from the usual amount ·0062.

On account of the special importance attaching to these observations the values of observed index—probable index

for all Gifford's values quoted by Lamplough and Mathews are tabulated below. By 'observed index' is meant the value obtained by regarding all the figures published in Lamplough and Mathews' Table I as an exact representation of the observations, and by 'probable index' the value derived from a curve constructed in the way already described. An examination of this table shows that the departures are not methodical and that there is therefore no reason for doubting the general conclusions.

Differences, observed—probable; unit 10^{-6}.

Number of glass	*A′*	*B′*	*Li*	*C*	*D*	A	*Hg*	*E*	*F*	Φ	*Cd*	*G′*	*Hg*	Value of β
1	− 28	− 3	+ 6	+ 5	+ 9	− 12	+ 8	− 8	− 3	+ 18	+ 2	− 37	+ 16	·00630
3	− 18	+ 9	+ 11	+ 9	+ 4	− 10	− 7	− 13	+ 1	+ 7	+ 4	+ 4	− 7	·00620
4	− 12	− 12	+ 13	+ 16	− 13	− 19	+ 7	+ 16	− 10	− 12	+ 36	− 40	− 16	·00643
5	− 9	+ 10	+ 9	− 17	− 4	+ 9	+ 1	− 9	+ 22	− 1	+ 11	− 1	− 41	·00620
6	− 5	+ 3	+ 19	− 21	+ 3	− 18	+ 7	+ 2	+ 12	− 20	+ 17	+ 10	− 18	·00620
8	− 18	+ 22	− 17	+ 31	0	− 13	− 12	+ 21	− 3	+ 6	− 9	− 22	+ 2	·00636
9	+ 26	− 2	− 1	0	+ 10	− 3	+ 10	− 15	− 7	+ 7	+ 20	0	− 39	·00609
10	+ 30	− 26	− 14	+ 12	+ 25	− 14	+ 17	− 11	− 17	+ 14	− 3	+ 4	− 38	·00628
11	+ 2	+ 7	+ 3	− 7	− 17	+ 3	+ 3	+ 2	+ 10	− 1	+ 20	− 11	− 21	·00620
12	0	+ 10	+ 20	+ 24	− 8	+ 8	− 30	+ 9	− 4	+ 3	+ 22	− 11	− 16	·00620
13	+ 8	− 15	+ 7	− 9	− 4	+ 2	+ 8	− 2	− 5	+ 19	+ 5	− 2	− 13	·00620
14	− 14	− 1	− 20	+ 9	+ 7	+ 9	− 7	− 18	− 4	− 1	+ 3	− 41	+ 9	·00652
15	− 45	+ 20	− 36	+ 34	− 8	− 2	+ 3	+ 3	+ 15	+ 6	+ 6	− 1	− 42	·00620
16	+ 1	− 4	+ 7	+ 8	− 23	− 10	− 16	+ 14	+ 9	+ 40	+ 2	− 22	− 16	·00620
17	+ 14	− 16	+ 10	+ 10	+ 1	− 10	− 9	+ 20	+ 10	− 18	+ 8	− 2	− 17	·00643
18	+ 16	+ 11	− 21	+ 2	− 15	+ 6	− 4	− 12	+ 19	+ 20	+ 5	+ 1	− 18	·00620
22	− 13	− 6	+ 11	+ 15	+ 4	− 9	− 18	− 6	+ 8	+ 15	+ 10	− 13	− 12	·00620
24	− 1	+ 4	+ 10	− 18	− 2	+ 20	− 14	− 3	+ 3	+ 12	0	− 14	+ 6	·00620
25	+ 5	+ 3	− 11	− 5	− 9	− 5	0	+ 23	+ 22	− 4	+ 5	− 4	+ 8	·00620
26	+ 11	− 3	− 6	− 5	+ 4	− 2	+ 9	− 4	+ 8	+ 1	− 3	− 21	+ 26	·00620
27	− 7	− 9	+ 5	+ 10	+ 7	+ 15	− 7	+ 3	+. 1	− 8	− 2	− 4	+ 16	·00620
28	− 13	− 17	+ 5	+ 18	+ 13	+ 8	− 14	+ 4	− 18	− 18	− 13	− 9	+ 29	·00620
29	− 5	+ 20	− 9	− 9	− 1	+ 3	+ 10	+ 4	+ 2	+ 31	+ 8	− 32	− 9	·00620
31	+ 27	− 11	− 5	+ 5	+ 19	+ 5	− 15	+ 2	− 18	− 22	+ 42	− 3	− 20	·00620
32	− 16	+ 1	+ 9	0	− 15	+ 20	+ 13	− 12	+ 17	+ 11	+ 12	− 27	− 20	·00620
33	− 17	− 10	+ 5	+ 12	+ 14	+ 22	− 1	+ 6	− 6	− 8	− 4	− 16	+ 21	·00620
35	+ 4	+ 2	− 6	+ 4	− 9	+ 4	− 9	+ 3	− 5	+ 27	+ 15	− 5	− 13	·00626
36	− 17	− 6	− 16	0	+ 10	+ 2	+ 11	+ 21	0	− 10	− 21	− 31	− 20	·00620
37	− 5	+ 2	− 14	+ 8	+ 13	+ 9	+ 16	+ 10	− 7	− 10	− 2	− 7	+ 20	·00620
38	+ 4	− 2	− 6	+ 4	− 6	+ 4	− 10	+ 10	− 16	+ 9	+ 15	0	− 34	·00626

An inspection of the relative number of plus and minus signs in the columns of the above table shows no serious departure from probable numbers except at the violet end. In the last three columns the numbers of plus and of minus signs are

	Cd	*G'*	*Hg*
Plus	21	4	10
Minus	8	24	20

and further investigation in this region is of importance. It is not unlikely that this irregularity may in part be due to the special difficulties involved in measurements in this region of the spectrum under normal conditions. In using the dispersion formula the figures in ordinary use for wave lengths are not sufficiently accurate, and the most accurate values known should be used.

It has been seen that since the value of β is practically constant for glasses in general, special difficulty is found in removing the secondary spectrum from lens combinations. Recourse must therefore be had to other transparent media, and crystals of the cubic system are of special interest. In Fig. 1 the points representing the most important of these have been added, and it will be seen that fluorite lies nearly on the line typical of normal glasses. This indicates that this substance is the most suitable for combination with glasses for the removal of the secondary spectrum, for if the various materials used are selected by seeing that they are represented in the diagram by collinear points they all obey the same relation of the form

$$P(\mu_1 - \mu_2) + Q(\mu_3 - \mu_4) + R(\mu_5 - \mu_6) = 0,$$

which shows that if the system satisfies two colour conditions making the powers equal for wave lengths 1 and 2, and also for 3 and 4, the powers will necessarily be equal in addition for 5 and 6. As is well known, and this difference is vital, fluorite does not conform to the relation between dispersion and deviation characteristic of glass, so that when a formula of the kind already considered applicable to fluorite and the selected glasses has been constructed the values of β will differ appreciably. Investigation shows that if glasses are selected by the method suggested, all valuable combinations can be noted in a minute or two, and the best of these lead to systems that are exceptionally free from chromatic variations of the aberrations in addition to being free from secondary spectrum. The methods of calculating such objectives and of finding the best order in which the media should be arranged are described elsewhere*. The most important case involves the use of three different materials, and the total powers for the lenses of each of these are given by

$$\frac{\kappa_1}{\nu_1(\beta_2\nu_2 - \beta_3\nu_3)} = \frac{\kappa_2}{\nu_2(\beta_3\nu_3 - \beta_1\nu_1)} = \frac{\kappa_3}{\nu_3(\beta_1\nu_1 - \beta_2\nu_2)}$$

$$= \frac{-\kappa}{\beta_1\nu_1(\nu_2 - \nu_3) + \beta_2\nu_2(\nu_3 - \nu_1) + \beta_3\nu_3(\nu_1 - \nu_2)}.$$

* *Trans. Opt. Soc.* 22 (1920–21), 111.

If the two glasses, 2 and 3, have the same value of β, and for the crystal the increase in β is denoted by δ, so that $\beta_1 = \beta + \delta$, the equations become

$$\kappa_1 = -\kappa\beta/\delta,$$
$$\kappa_2 = \kappa\nu_2\{1 + (\nu_1 - \nu_3)\,\beta/\nu_1\delta\}/(\nu_2 - \nu_3),$$
$$\kappa_3 = -\kappa\nu_3\{1 + (\nu_1 - \nu_2)\,\beta/\nu_1\delta\}/(\nu_2 - \nu_3),$$

and reasonable curvatures are secured if δ approaches β in magnitude. From the low dispersion of fluorite it will be used for a positive component in a positive combination, so that δ is negative, and the tendency exhibited by these equations is towards an objective with positive lenses of fluorite and of the more dispersive glass, and a correcting negative of the glass of less dispersion. If β is independent of the variety of glass the power of the fluorite lenses is constant; and the powers of the glasses are controlled by securing a sufficient ν difference just as with an ordinary doublet. This is at once evident if the glasses are regarded as an uncorrected combination of given power introduced to compensate the dispersion of a known fluorite lens.

THE ABERRATION THEORY OF THIN OBJECTIVES CONSTRUCTED WITH THREE OR MORE KINDS OF GLASS

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Abstract. General formulae are developed for the rapid calculation of the aberration coefficients of thin objectives, and special consideration is given to cemented objectives constructed from several different glasses. The most important problem is the knowledge of the influence of the order of the lenses on the spherical aberration in an image (inverted) at magnification unity. With only one lens of each kind of glass this gives three cases to be considered with three glasses, twelve cases with four glasses, sixty cases with five glasses, and in general $\frac{1}{2}n!$ cases with n glasses. A formula is given for the mean coefficient of all cases, the evaluation of which shows whether an arrangement giving a high, a medium, or a low value of the coefficient is required for the purpose in view. Rules are formulated which indicate where in the series such values are likely to be found.

The author has shown previously* that three quantities suffice to determine the aberrations of any thin lens, and has given convenient expressions from which they may in any case be calculated. These are susceptible of transformation into various shapes for dealing with different classes of compound lenses, and they have been discussed in detail in their application to multiple cemented lenses where only two kinds of glass are used. Such lenses meet all requirements consistent with negligible thicknesses when light of a single wave length is considered, but with known transparent substances this limitation involves a less perfect correction of the chromatic aberrations than is attainable if three or more different materials are introduced. This increase in the number of the media introduces much complexity into the general investigation of lenses of this class, for after suitable glasses for the attainment of the required corrections for colour have been selected many different systems may be constructed from them by arranging the glasses in the various possible orders. It is not obvious what order will be best for a given purpose, and in the absence of a suitable rule† to indicate the best arrangement much time may be absorbed in finding the effect of varying the order. The object of the present paper is to present the general formulae in a shape which will show at once which

* *Proc. Phys. Soc.* 27 (1915), 485.

† The author is under the impression that he has seen a statement attributed to Steinheil that the glasses should be arranged with their refractive indices continuously increasing or continuously decreasing, but has been unable to verify this.

of the possible alternative arrangements should be chosen, and to give with the minimum of labour full information concerning the corrections possible with the lenses placed in various orders. The results so obtained will, as in other cases, be directly applicable to the construction of well corrected lenses of appreciable apertures by introducing suitable thicknesses.

The system is assumed to consist of n thin lenses placed in contact, each distinguished by a numerical suffix. In the standard case the suffixes $1, 2, 3, \ldots p \ldots n$ indicate the order in which light travelling from the object to the image meets the components. The power of, say, the pth component lens will be denoted by κ_p, its refractive index by μ_p, the sum of the curvatures of its refracting surfaces by $2\rho_p$, the difference and the sum of the curvatures of its unit surfaces by $\varpi_p\kappa_p$ and $\beta_p\kappa_p$ respectively, and its spherical aberration coefficient for negative unit magnification by $\frac{1}{4}\gamma_p$. For a single thin lens

$$\varpi = \frac{1}{\mu} \qquad \ldots\ldots(1),$$

$$\beta = \frac{2\rho}{\kappa}(1 + \varpi) \qquad \ldots\ldots(2),$$

$$\gamma = \frac{1}{(1 - \varpi)^2} + \frac{4\rho^2}{\kappa^2}(1 + 2\varpi) \qquad \ldots\ldots(3),$$

and for any combination of thin lenses in contact

$$\varpi_{1,n}\,\kappa_{1,n} = \sum_1^n \varpi_p\kappa_p \qquad \ldots\ldots(4),$$

$$\beta_{1,n}\,\kappa^2_{1,n} = \sum_1^n \{\beta_p\kappa_p - \varpi_p(\kappa_{1,p} - \kappa_{p,n})\}\,\kappa_p \qquad \ldots\ldots(5),$$

$$\gamma_{1,n}\kappa^3_{1,n} = \sum_1^n \{\gamma_p\kappa_p{}^2 - 4\beta_p\kappa_p(\kappa_{1,p} - \kappa_{p,n}) + (3 + 2\varpi_p)(\kappa_{1,p} - \kappa_{p,n})^2\}\,\kappa_p \qquad \ldots\ldots(6),$$

where $\kappa_{p,q}$ denotes $\kappa_p + \kappa_{p+1} + \ldots + \kappa_{q-1} + \kappa_q$, and $\varpi_{1,n}$, $\beta_{1,n}$, $\gamma_{1,n}$ have the same meanings for the combination as ϖ_p, β_p, γ_p bear for the single lens. These formulae follow at once from the well-known magnification laws for the transference of aberrations through subsequent lenses, and from the condition for freedom from spherical aberration in a thin lens for magnification m

$$\gamma - 4\beta M + (3 + 2\varpi)M^2 = 0 \qquad \ldots\ldots(7)$$

and for freedom from coma with the stop close to the lens

$$\beta - (2 + \varpi)M = 0 \qquad \ldots\ldots(8),$$

where

$$M = \frac{1 + m}{1 - m}$$

if it is noted that

$$\sum_1^n (\kappa_{1,p} - \kappa_{p,n})\,\kappa_p \equiv 0.$$

When the value of β given by (2) is substituted in the right side of (5), the resulting equation* is

$$\beta\kappa^2 = \Sigma\{2\rho_p(1+\varpi_p) - \varpi_p(\kappa_{1,p} - \kappa_{p,n})\}\kappa_p \qquad \ldots\ldots(9)$$

from which it is seen that an increase in each ρ by the same amount ρ' leads to the new value β', where

$$\beta' = \beta + \frac{2\rho'}{\kappa}(1+\varpi) \qquad \ldots\ldots(10)$$

in virtue of (4). This is a generalised form of (2), and it would be possible to secure exact agreement by selecting as the reference form of the compound lens that configuration in which the curvatures of the unit surfaces are equal and opposite. It is preferred, however, to select the form in which the expression for γ corresponds to (3), the spherical aberration for unit magnification then being a minimum. The value of β corresponding to this configuration is always small.

Substituting from (2) and (3) in (6)

$$\gamma\kappa^3 = \Sigma\left\{\left(\frac{\kappa_p}{1-\varpi_p}\right)^2 + 4\rho_p^2(1+2\varpi_p) - 8\rho_p(1+\varpi_p)(\kappa_{1,p} - \kappa_{p,n}) + (3+2\varpi_p)(\kappa_{1,p} - \kappa_{p,n})^2\right\}\kappa_p \quad \ldots(11),$$

and increasing each ρ by ρ' as before gives the new value γ' where

$$\gamma' = \gamma + \frac{4\rho'^2}{\kappa^2}(1+2\varpi) + \frac{8\rho'}{\kappa^3}\Sigma\{\rho_p(1+2\varpi_p) - \varpi_p(\kappa_{1,p} - \kappa_{p,n})\}\kappa_p \quad \ldots(12)$$

Comparison of the coefficient of ρ' with (9) shows that the standard configuration is attained by imposing the condition

$$\beta\kappa^2 = \Sigma\rho_p\kappa_p \qquad \ldots\ldots(13).$$

A simple method of satisfying this condition is to take any set of values of the ρ's consistent with the geometrical conditions to be observed, and adopt in place of ρ_p the value

$$\rho_p - \frac{1}{(1+2\varpi)\kappa}\sum_1^n\{\rho_p(1+2\varpi_p) - \varpi_p(\kappa_{1,p} - \kappa_{p,n})\}\kappa_p \qquad \ldots\ldots(14),$$

which obviously causes the sum in (12) to vanish. It follows from (13) that in the standard configuration β may be calculated from

$$\beta_0(1+2\varpi)\kappa^2 = \sum_1^n(\varpi - \varpi_p)(2\rho_p - \kappa_{1,p} + \kappa_{p,n})\kappa_p \qquad \ldots\ldots(15),$$

where any set of values for the ρ's is used. The small range of refractive indices available restricts the values of all the individual ϖ's within narrow limits, and it follows that in all but very extreme systems the first factor in each term of this sum is small, and consequently the value of β_0 is small. It is convenient to distinguish the β and γ of the lens in its standard form by the addition of the suffix o as in equation (15).

* In the absence of any suffix to κ, ϖ, β, γ the double suffix $1, n$ is to be understood hereafter.

If equation (11) is treated similarly the result will be

$$\gamma_0\kappa^3 = \sum_1^n \left\{\left(\frac{\kappa_p}{1-\varpi_p}\right)^2 + 4\rho_p{}^2(1+2\varpi_p) - 8\rho_p(1+\varpi_p)(\kappa_{1,p}-\kappa_{p,n}) + (3+2\varpi_p)(\kappa_{1,p}-\kappa_{p,n})^2\right\}\kappa_p - \frac{4}{(1+2\varpi)\kappa}\left[\sum_1^n\{\rho_p(1+2\varpi_p) - \varpi_p(\kappa_{1,p}-\kappa_{p,n})\}\kappa_p\right]^2 \quad \ldots\ldots(16),$$

and it can be seen by inspection that if every ρ is increased by the same constant the expression on the right is unaffected in value. Equations (15) and (16) may be simplified by the substitution

$$2\rho_p = \kappa_{1,p} - \kappa_{p,n} + \sigma_p \quad \ldots\ldots(17),$$

where σ_p is a new variable fixing the shape of the component lens. The new expressions are

$$\left.\begin{aligned} \beta_0 k\kappa &= \sum_1^n(\varpi-\varpi_p)\sigma_p\kappa_p \\ \text{or} \quad 2\beta_0\kappa &= \frac{1}{\kappa}\sum_1^n\sigma_p\kappa_p - \frac{1}{k}\sum_1^n\sigma_p k_p \end{aligned}\right\} \quad \ldots\ldots(18),$$

and

$$\gamma_0\kappa^3 = \sum_1^n\{j_p{}^2\kappa_p + \sigma_p{}^2k_p - 2\sigma_p(\kappa_{1,p}-\kappa_{p,n})\kappa_p\} - \frac{1}{k}\left[\sum_1^n\sigma_p k_p\right]^2 \quad \ldots\ldots(19),$$

where

$$j_p = \frac{\kappa_p}{1-\varpi_p}$$

and

$$k_p = \kappa_p(1+2\varpi_p),$$

and the absence of a suffix as before denotes summation of the individual values throughout the system. Since the values of β_0 and γ_0 are unaffected by the addition of a constant to all the ρ's, or what is the same thing to the σ's, these expressions are functions of their differences only. This may be exhibited in many ways, as by giving particular values to the additive constant, or by rearrangement of the terms. For instance

$$\beta_0 k\kappa = \Sigma\{(\sigma_p - \sigma_{p+1})\sum_1^p(\varpi-\varpi_p)\kappa_p\}$$

and

$$\gamma_0\kappa^3 = \Sigma\left\{j_p{}^2\kappa_p + \frac{1}{k}k_pk_q(\sigma_p-\sigma_q)^2 + 2(\sigma_p-\sigma_{p+1})\kappa_{1,p}\kappa_{p+1,n}\right\} \quad \ldots(20),$$

where in the term involving the suffix q each pair, p, q, is taken once.

Since the principal problem is the determination of γ_0 it is important to obtain convenient expressions for the σ differences. These depend on the known lens constant and on the air gaps prescribed between successive components. Suppose that the curvature of the second surface of lens p is to exceed that of the first surface of lens $p+1$ by ϵ_p. Then

$$\rho_p - \frac{\kappa_p}{2(\mu_p-1)} = \epsilon_p + \rho_{p+1} + \frac{\kappa_{p+1}}{2(\mu_{p+1}-1)},$$

or from (17)

$$\sigma_p - \sigma_{p+1} = 2\epsilon_p + j_p + j_{p+1} \quad \ldots\ldots(21)$$

This is satisfied by writing

$$-\sigma_p = \epsilon_{1,p-1} - \epsilon_{p,n-1} + j_{1,p} - j_{p,n}$$

which also involves

$$\sigma_1 + \sigma_n + j_1 - j_n = 0$$

the condition that the curvatures of the first and last surfaces of the system should be equal and opposite. It follows from (14) or (19) that in the standard form the sum of these extreme curvatures is

$$\left.\begin{aligned} &\frac{1}{k}\sum_1^n (\epsilon_{1,p-1} - \epsilon_{p,n-1} + j_{1,p} - j_{p,n})\, k_p \\ \text{or}\quad &-\frac{1}{k}\left[\sum_1^{n-1} \epsilon_p (k_{1,p} - k_{p+1,n}) + \sum_1^n j_p (k_{1,p} - k_{p,n})\right] \end{aligned}\right\} \quad \ldots\ldots(22)$$

and that for this configuration

$$\begin{aligned} -\sigma_p k = {} & 2\sum_1^{p-1} \epsilon_p k_{1,p} - 2\sum_p^{n-1} \epsilon_p k_{p+1,n} \\ & + \sum_1^{p-1} j_p (k_{1,p} + k_{1,p-1}) - \sum_{p+1}^{n} j_p (k_{p,n} + k_{p+1,n}) + j_p (k_{1,p} - k_{p,n}) \end{aligned}$$

a value for which, as may be readily verified,

$$\Sigma \sigma_p k_p = 0.$$

Substituting from (21) in (20) gives

$$\begin{aligned} \gamma_0 \kappa^3 = \Sigma \Big\{ & j_p{}^2 \kappa_p + \frac{1}{k} k_p k_q (2\epsilon_{p,q} + j_{p,q-1} + j_{p+1,q})^2 \\ & + 2(2\epsilon_p + j_p + j_{p+1})\, \kappa_{1,p} \kappa_{p+1,n} \Big\} \quad \ldots(23). \end{aligned}$$

The second and third groups of terms either involve a single j with two κ's, or where there are two j's they are associated with k's and not κ's. The first term may be made to agree with these in form using the identity

$$3 j_p \kappa_p = j_p k_p + 2\kappa_p{}^2.$$

If this substitution is made the expression for γ_0 may take the form

$$\begin{aligned} \gamma_0 \kappa^3 = {} & \frac{1}{k} [\Sigma \{ j_p (k_{1,p} + k_{1,p-1}) + 2\epsilon_p k_{1,p} \}] [\Sigma \{ j_q (k_{q,n} + k_{q+1,n}) + 2\epsilon_q k_{q+1,n} \}] \\ & + \Sigma j_p (\kappa_{1,p} + \kappa_{1,p-1})(\kappa_{p,n} + \kappa_{p+1,n}) + 4\Sigma \epsilon_p \kappa_{1,p} \kappa_{p+1,n} \\ & - \tfrac{1}{3}\Sigma j_p \kappa_p{}^2 - \tfrac{8}{3}\Sigma j_p{}^2 \kappa_p \qquad \ldots\ldots(24), \end{aligned}$$

where the product of the series is to be interpreted to mean that when like terms such as $j_p j_q$ and $\epsilon_p \epsilon_q$ with $p = q$ are not involved, the coefficient from the earlier sum is to be associated with the element occurring first in the system, and that from the second sum with that coming subsequently, and that this coefficient is to be doubled. Thus if $p < q$ the coefficient of $j_p j_q$ is

$$\frac{2}{k}(k_{1,p} + k_{1,p-1})(k_{q,n} + k_{q+1,n}),$$

and that of $j_p \epsilon_p$ is

$$\frac{4}{k}(k_{1,p} + k_{1,p-1})\, k_{q+1,n}.$$

It should be particularly noted that since j_p precedes ϵ_p, this formula for the term in $j_p\epsilon_q$ applies when $p = q$. Equation (24) exhibits clearly the effect of varying any particular air gap shape, but in the applications to be investigated here these will all be assumed to be zero.

The main problem to be discussed is the effect of change of order of the elements upon γ_0 when the lenses are arranged to form a cementable group. When there are n lenses, these can be arranged in $n!$ ways, but since γ_0 does not depend on the direction in which light travels there are only $\frac{1}{2}n!$ cases to be investigated. In the two arrangements here paired as a single case the values of β_0 are equal in magnitude but opposed in sign so that the mean value for β_0, and similarly for the curvature sum of the end surfaces, in the entire series must be zero. In the case of these quantities therefore no information of value is to be derived from finding mean values unless some restrictions on the extent of the series summed is made. There is, however, here no difficulty in finding the effect of any variation, and it is easy to determine the effect of changes within limited groups. For instance, if lenses $p + 1, \ldots n$ are removed from the end of the system and put at the beginning with an air gap ϵ_n between lenses n and 1, the sum of the end curvatures is increased by

$$2j_{p+1,n} + \epsilon_{p,n-1} + \epsilon_{p+1,n} - \frac{2(j+\epsilon)}{k} k_{p+1,n},$$

and the increase in $\beta_0 k\kappa$ is

$$2(j+\epsilon)(\varpi - \varpi_{p+1,n})\kappa_{p+1,n}.$$

When all possible changes of order are made in the component lenses the expressions for γ_0 in any particular case will be found to contain exactly one-third of all the variable terms which will be present in the total. For example in equation (23) the coefficient of $2j_p$ in the last sum is

$$\kappa_{1,p-1}\kappa_{p,n} + \kappa_{1,p}\kappa_{p+1,n}$$

or

$$\kappa\kappa_p - \kappa_p^2 + 2\kappa_{1,p-1}\kappa_{p+1,n}.$$

The variable terms are thus $4j_p\kappa_{1,p-1}\kappa_{p+1,n}$, the particular ones present depending on the order of the components. The number of terms of this type in any particular case is

$$4\{0(n-1) + 1(n-2) + 2(n-3) + \ldots + (n-2)1 + (n-1)0\}$$

or $4\dfrac{n(n-1)(n-2)}{6}$. The total terms of this kind are evidently given by

$$2\Sigma j_p\{(\kappa - \kappa_p)^2 + \kappa_p^2 - \Sigma\kappa_p^2\}$$

and the number of terms here is

$$2n\{(n-1)^2 + 1 - n\} = 2n(n-1)(n-2)$$

or three times the number present in the particular case. It follows that in the mean value of $\gamma_0\kappa^3$ the contribution of these unsymmetrical terms will be

$$\tfrac{2}{3}\{j(\kappa^3 - \Sigma\kappa_p^3) - 2\kappa\Sigma j_p\kappa_p + 2\Sigma j_p\kappa_p^2\}.$$

The symmetrical terms, which are the same in all cases, are

$$2\kappa\Sigma j_p\kappa_p - 2\Sigma j_p\kappa_p^2,$$

and the first term when divided yields

$$\tfrac{2}{3}\Sigma j_p\kappa_p^2.$$

The total contribution to the mean value of terms of this kind is therefore

$$\tfrac{2}{3}\{j\kappa^2 - j\Sigma\kappa_p^2 + \kappa\Sigma j_p\kappa_p\}.$$

The terms involving the k's can be treated in a similar way, but the investigation is more complicated as the terms present fall into three dissimilar groups each of which requires separate investigation. The result reached is that the mean value of γ_0 may be calculated from

$$\text{Mean } \gamma_0\kappa^3 = \tfrac{2}{3}\{j\kappa^2 - j\Sigma\kappa_p^2 + \kappa\Sigma j_p\kappa_p\} + \frac{1}{3k}\{(j^2 + \Sigma j_p^2)(k^2 - \Sigma k_p^2) + (\Sigma j_p k_p)^2\} \quad \ldots\ldots(25).$$

The evaluation of this mean shows whether for a given purpose an arrangement giving a high, a medium, or a low value of γ_0 is required. Where the lens is required to be self-corrected for spherical aberration and coma, an arrangement giving a value of γ_0 near zero is required. The general tendency for lenses corrected for colour is to have distinctly negative values for γ_0, so that the selection in these cases will be to find the arrangement for which γ_0 exceeds the mean. This negative tendency can be seen by considering the terms $\kappa^2 - \Sigma\kappa_p^2$ and $k^2 - \Sigma k_p^2$, which multiply factors necessarily positive. Suppose there are n lenses in all, of which $n - r$ are positive and r negative. There will be $(n-r)(n-r-1) + r(r-1)$ positive terms, and $2(n-r)r$ negative terms. If the positive lenses are each of power $p\kappa$ and the negative each of power $-q\kappa$

$$\kappa^2 - \Sigma\kappa_p^2 = \kappa^2\{1 - (n-r)p^2 - rq^2\} = \kappa(1 - p + q - npq)$$

since

$$1 = (n-r)p - rq.$$

This expression indicates clearly the negative tendency of the majority of the terms in $\gamma_0\kappa^3$. If $p\kappa$ and $-q\kappa$ are only the mean values of the powers of the positive and negative lenses respectively, there being inequalities among them, the expression $\kappa^2 - \Sigma\kappa_p^2$ itself shows that the negative tendency is increased.

One of the methods by which the influence of changes of order can be exhibited is to find an expression for the difference between the value of γ_0 for an assigned order and the mean value (25). This offers no difficulty, but an alternative treatment based on the equations used for numerical calculations will be considered here.

The equation for γ_0, say (23), may be rearranged in the form

$$\gamma_0\kappa^3 = \Sigma(2\kappa - 2\kappa_p + j_p)j_p\kappa_p + \frac{1}{k}\Sigma(j_p + j_q)^2 k_p k_q + 4\Sigma j_p\kappa_{1,p-1}\kappa_{p+1,n} + \frac{4}{k}\sum_{p<q} j_{p,q}j_{p+1,q-1}k_p k_q \ldots(26).$$

The terms in the first line are independent of the order, and attention need therefore only be given to the terms of the second line. The second of these sums is equal to

$$\frac{4j}{k}\,\Sigma j_p k_{1,p-1} k_{p+1,n} - \frac{4}{k}\,\Sigma\,(j_p k_q j_r k_s + k_p j_q k_r j_s). \qquad p<q<r<s$$

The variations of the quantity in the second sum are of minor importance. They do not vary at all for any system in which the components are in the same cyclical order, and their departure from the mean in other cases is small. Thus considering the terms in which the suffixes are 1, 2, 3, 4

$$\begin{aligned}3\,(j_1k_2j_3k_4 + k_1j_2k_3j_4) - \Sigma k_p j_q k_r j_s \\ = j_1j_2j_3j_4\,\{&(\varpi_1-\varpi_2)(\varpi_3-\varpi_4)(2\varpi_1+2\varpi_2-1)(2\varpi_3+2\varpi_4-1) \\ &-(\varpi_1-\varpi_4)(\varpi_2-\varpi_3)(2\varpi_1+2\varpi_4-1)(2\varpi_2+2\varpi_3-1)\}. \\ &(p,q,r,s) = (1,2,3,4)\end{aligned}$$

The terms in the large bracket will not exceed

$$3\,\{(\varpi_1-\varpi_2)(\varpi_3-\varpi_4)-(\varpi_1-\varpi_4)(\varpi_2-\varpi_3)\}$$

a second order quantity which may be neglected.

The terms to be considered are thus

$$\Sigma j_p\left(\kappa_{1,p-1}\kappa_{p+1,n} + \frac{j}{k}\,k_{1,p-1}k_{p+1,n}\right),$$

and the variations depend only on the relative magnitudes of j_p, k_p, κ_p and on the sign of the power. The influence of changes of order is apparent if it is noted that for any component

$$|\,j\,|>|\,k\,|>|\,\kappa\,|$$

and that these inequalities are more pronounced the greater the value of ϖ, that is the smaller the refractive index, since this must lie between 1 and 2. Any term in the sum has thus its greatest numerical value when the central component has the greatest value of ϖ, and its least numerical value when the lens of greatest refractive index is in the centre. Thus with three glasses, the extreme values of γ_0 will be obtained by putting a component of extreme index between the other two. If $\kappa_1\kappa_2\kappa_3$ has the same sign as κ, the lens of least index should be in the centre to attain the highest value of γ_0, and the lens of highest index to attain the lowest value. If $\kappa_1\kappa_2\kappa_3$ differs in sign from κ, the lowest γ_0 results from placing the least index centrally, the highest γ_0 from placing the greatest index there. If the lens is composed of one fluorite component and two kinds of glass, the glass of greatest index will usually be positive to produce a mean dispersion curve for the positive component which the remaining negative lens of medium index can balance. The fluorite and the negative lens will therefore be the outside component in the lens best suited for ordinary conditions. These results may be made apparent in a difference equation. Let γ_0 correspond to the order 1, 2, 3 and γ_0' to the order 1, 3, 2. Then from (26)

$$\begin{aligned}(\gamma-\gamma_0')\,\kappa^3 &= \kappa_1 j_2\kappa_3 - \kappa_1\kappa_2 j_3 + \frac{j}{k}\,(k_1 j_2 k_3 - k_1 k_2 j_3) \\ &= (\varpi_2-\varpi_3)\left\{\kappa_1 + \frac{j}{k}\,k_1\,(2\varpi_2+2\varpi_3-1)\right\} j_2 j_3.\end{aligned}$$

When there are four glasses the investigation is necessarily more complicated, for it will usually be the case that some of the terms in the difference from the mean value will be positive, and others negative. The exact determination of the position in order of magnitude of γ_0 of any given arrangement thus necessarily involves a knowledge of the relative magnitudes of the powers of the components and of the differences in the ϖ's. When the power of one component is very small the system must as regards the remaining components be arranged according to the indications given for three glasses. In other cases a fair indication may be gathered by considering how many of the terms to be added to the mean value are positive, and how many are negative. Since the influence of both the κ and the k terms lies in the same direction, the former alone need be considered. Let the symbol $p\,(qr)$ or $(qr)\,p$ denote $\kappa_p j_q j_r (\varpi_q - \varpi_r)$. Then apart from a numerical coefficient the additional κ terms, on simplifying by the identity

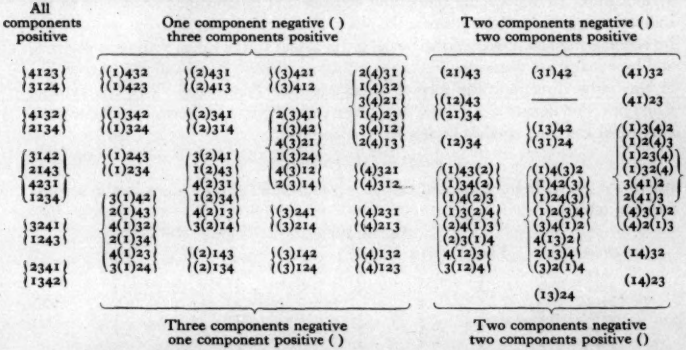
$$p\,(qr) + (qp)\,r = q\,(pr)$$

where $p < q < r$, become for the various arrangements those shown in the accompanying table.

	Additional terms							
Order	(12) 3	1 (23)	(12) 4	1 (24)	(13) 4	1 (34)	(23) 4	2 (34)
1234	− 1	1	− 1	1	− 1	1	− 1	1
2341	− 1	− 2	− 1	− 2	− 1	− 2	− 1	1
3412	2	1	2	1	− 1	− 2	− 1	− 2
4123	− 1	1	2	1	2	1	2	1
1324	− 1	− 2	− 1	1	− 1	1	2	1
3241	− 1	1	− 1	− 2	− 1	− 2	2	1
2413	2	1	− 1	− 2	2	1	− 1	− 2
4132	− 1	− 2	2	1	2	1	− 1	1
1243	− 1	1	− 1	1	− 1	− 2	− 1	− 2
2431	− 1	− 2	− 1	− 2	− 1	1	− 1	− 2
4312	2	1	2	1	− 1	1	− 1	1
3124	2	1	− 1	1	2	1	2	1

If it is assumed that $\varpi_1 > \varpi_2 > \varpi_3 > \varpi_4$, the signs can be determined from the above table on taking the signs of the components concerned in the addition into account. A general indication of the influence of order may be derived on grouping according to the net additions, neglecting the difference in magnitude of the component powers, and assuming that the increments in ϖ between successive components in their natural order are equal. This is equivalent to giving double weight to the two central columns of the above table. It will be found that a number of groups are formed with a common difference of six additional terms between one group and the next. The order within the groups may be differentiated by assuming that all the positive lenses are of equal power, and likewise all the negative lenses of equal power, the power of any positive lens exceeding that of any negative lens except when the system contains only one negative component, when a positive lens is assumed to be less powerful than the negative. In the following table the numbers of the negative components are placed within round brackets. The large brackets on the left indicate the groups, and the small brackets on the right the subgroups. It is only necessary to give half the total number of cases, since if the sign of each lens is changed it is only necessary to read the columns in the reverse

order*. The table is arranged to show the order of the components when all are positive, when 1, 2, 3, 4 in turn are negative and the rest positive, and when 1 and 2, 1 and 3, 1 and 4 are negative and the others positive, the value of γ_0 tending to be greatest for the first arrangement in the column, and falling step by step until it tends to be least for the arrangement given last. It is assumed throughout that the power of the whole combination is positive.



It will be noted that the only cases where brackets occur on the right concern arrangements of the types *pqrs*, *sqrp*, *rspq*, *qspr*. Differentiation between these depends upon the relative powers of the components and in any given case can be determined at once. From this table it is seen that, apart from exceptional cases mentioned below, extreme values of γ_0 are attained as follows:

(1) By grouping all the positive lenses together and all the negative lenses together;

(2) By arranging the lenses of like sign in the order of their refractive indices;

(3) For a high γ_0 the negative lens of lowest index should be in contact with the positive lens of highest index: for a low γ_0 the negative lens of highest index should be placed next to the positive lens of lowest index.

The exceptional extreme cases occur when for a high γ_0 the indices of all negative lenses exceed that of any positive lens, and for a low γ_0 the indices of all the positive lenses exceed that of any negative lens. In the former case the positive group, and in the latter the negative group, occupies a central position, or alternatively both are divided.

It will be observed that the position of the group to which any arrangement belongs depends upon the number of steps by which the arrangement departs from the extremes given by the above rules.

* When two components are positive and two negative, an exchange of signs reverses the order of the groups without affecting the order within the group. In this case therefore when the columns are read from the bottom upwards each group is to be read downwards.

It may be of interest to take an arbitrary example of each kind and see how nearly the order of the lists is followed. It would appear from the instances worked out below that the table indicates with very fair accuracy the actual order. It is evident that the formula for the mean γ_0 is of distinct value in indicating whether a given combination of glasses with assigned powers will give a γ_0 near the value desired. Even where the lenses need not be cemented it is useful to calculate its value, as for a number of reasons it is advantageous to keep the curvature difference of air gaps small.

Example I. One negative lens

Suffix	w	κ	k	j
1	·675	−1·3	−3·055	−4
2	·65	·7	1·61	2
3	·625	1·2	2·7	3·2
4	·6	·4	·88	1
1, 4	·5675	1	2·135	2·2

Mean γ_0 from (25) = 13·491

Order from table	γ_0	β_0	Sum of curvatures of end surfaces
1432	22·512	·1936	−1·0927
1423	19·929	·1498	− ·8604
1342	19·570	·1680	− ·9841
1324	19·598	·1382	− ·8436
1243	15·083	·1201	− ·7199
1234	14·272	·0944	− ·6112
3142	6·532	·0076	− ·0248
2143	9·005	·0746	− ·4108
4132	12·187	·1114	− ·6571
2134	8·051	·0490	− ·3021
4123	9·172	·0676	− ·4248
4213	5·985	·0222	− ·1157

Example II. Two negative lenses

Suffix	w	κ	k	j
1	·675	1·3	3·055	4
2	·65	− ·7	−1·61	−2
3	·625	1·2	2·7	3·2
4	·6	− ·8	−1·76	−2
1, 4	·6925	1	2·385	3·2

Mean γ_0 from (25) = 4·189

Order from table	γ_0	β_0	Sum of curvatures of end surfaces
4231	14·884	·0032	− ·0335
1432	11·093	·0083	− ·0235
1423	8·663	·0643	− ·2314
3421	9·829	·0099	− ·0200
1234	7·518	·0528	− ·1744
3412	7·716	·0716	− ·2969
2314	4·066	·2018	− ·7564
4312	5·661	·0088	− ·1023
3214	2·409	·1458	− ·5484
1342	4·133	·0710	− ·2180
3124	− 6·962	·2074	− ·8252
3142	−18·743	·2816	−1·0767

Example III. Three negative lenses

Suffix	w	κ	k	j
1	·675	− ·65	−1·5275	−2
2	·65	− ·35	− ·805	−1
3	·625	2·4	5·4	6·4
4	·6	− ·4	− ·88	−1
1, 4	·59375	1	2·1875	2·4

Mean γ_0 from (25) = 12·226

Order from table	γ_0	β_0	Sum of curvatures of end surfaces
4213	19·540	·2363	−·3269
2413	18·055	·2049	−·2926
4123	16·740	·2117	−·2891
1423	10·927	·1243	−·1829
4132	13·554	·1185	−·1758
2134	12·948	·2253	−·2923
1324	10·187	·1075	−·1413
1234	10·947	·1889	−·2606
1342	8·422	·1389	−·1755
1432	7·530	·0310	−·0695
2143	9·968	·1174	−·1863
1243	7·897	·0929	−·1486

A RECALCULATION OF THE TELESCOPIC OBJECTIVES OF STEINHEIL AND VOIT

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Read, 15th December, 1920.

The well-known work of Steinheil and Voit, one of the outstanding contributions to Optical Science from German workers, has lately become readily accessible to English readers through the translation by J. W. French which appears under the title *Applied Optics*. This translation repeats a number of errors, present in the original publication, which one of the present authors noted on going through the calculations some fifteen years ago. Recently these corrections have been redetermined by another method, and the results in both sets of calculations found in agreement. The scope of the calculations has been appreciably extended by including other rays than those considered by Steinheil and Voit and by the evaluation of a number of coefficients which represent the aberrations in an algebraic expansion in a series of aberrations of increasing orders. The corrections and a few of the simpler algebraic results are given in the tables which follow.

Table I.

Corrections to Constructional Data.

Form			Corrected value
A18a	...	...	$D_1 =$ 8·0
B11b	...	...	$R_4 =$ 1649·21
A10a	...	...	$R_8 = -$ 969·41

Using these corrected figures the results of tracing rays of the two colours at incident heights 0, 20, and 30 mm. are contained in the following tables. The first order aberration coefficients A, B, C, w are given in further tables, together with the paraxial power, and in addition the values of A and of the power when the curvatures are unaltered but all the surfaces touch one another on the axis.

The results derived from ray tracing have been given in full instead of by differences, to illustrate the systematic differences found in the figures as a whole, which may be due to the quotation by Steinheil and Voit of a smaller number of figures for the radii and separations than they used in their calculations. As a consequence of this discrepancy distinctly misleading conclusions may be drawn from the comparison of isolated additional ray-tracings with the published results.

The tracing of an intermediate ray gives further information of extreme importance to the objective maker. The extraordinary amount of the aberration for intermediate zones when the two components are strongly curved in opposite senses will be particularly noted.

Table II. $a'_{0,6}$.

Colour	*D*			*V*	
Form	$h = 0$	$h = 20$	$h = 30$	$h = 0$	$h = 30$
A 1a	670·533	664·634	670·497	670·506	679·650
A 2a	681·480	679·684	681·483	681·415	686·624
A 3a	687·464	686·715	687·477	687·432	690·898
A 4a	691·998	691·650	691·995	691·916	694·238
A 5a	702·585	702·535	702·610	702·558	703·272
A 6a	703·463	703·450	703·448	703·454	704·052
A 7a	709·106	709·100	709·117	709·078	709·493
A 8a	709·242	709·217	709·209	709·184	709·552
A 8a•	679·839	679·837	679·796	679·765	679·981
A 9a	710·297	710·301	710·315	710·263	710·683
A10a	664·498	664·632	664·517	664·483	664·479
A11a	713·433	713·446	713·442	713·439	713·901
A12a	713·789	713·799	713·784	713·765	714·313
A13a	716·723	716·711	716·690	716·706	717·253
A14a	717·867	717·863	717·872	717·871	718·516
A15a	719·999	719·961	720·007	719·966	720·749
A16a	720·637	720·583	720·637	720·655	721·475
A17a	721·507	721·427	721·509	721·481	722·376
A18a	725·036	724·801	725·047	725·016	726·238
A19a	736·292	734·254	736·303	736·268	739·084

Table III. $f_{0,6}$.

Colour	*D*			*V*	
Form	$h = 0$	$h = 20$	$h = 30$	$h = 0$	$h = 30$
A 1a	720·031	699·603	685·234	721·131	694·518
A 2a	719·934	708·481	697·339	720·671	702·661
A 3a	720·034	712·232	703·723	720·644	707·346
A 4a	719·974	714·473	708·128	720·411	710·577
A 5a	720·067	718·286	716·153	720·300	716·958
A 6a	720·019	718·477	716·528	720·247	717·264
A 7a	720·032	719·449	718·737	720·106	719·144
A 8a	720·040	719·464	718·759	720·083	719·131
A 8a•	720·088	719·814	719·445	721·042	720·644
A 9a	719·998	719·599	719·105	720·040	719·479
A10a	719·959	720·656	721·303	721·517	722·897
A11a	720·008	719·982	719·931	720·016	720·327
A12a	719·980	719·992	719·984	719·949	720·438
A13a	720·029	720·294	720·619	719·939	721·041
A14a	720·061	720·428	720·960	719·964	721·375
A15a	720·007	720·519	721·243	719·822	721·761
A16a	720·018	720·560	721·348	719·869	721·944
A17a	720·019	720·614	721·522	719·807	722·127
A18a	720·085	720·788	722·163	719·799	722·996
A19a	720·036	719·761	723·682	719·489	725·756

Table IV.

Form	$\kappa_0 \times 10^3$	A_0	$\kappa \times 10^3$	A	B	C	ϖ
A 1a	1·01623	−27·214	1·38883	34·081	84·834	142·382	0·575
A 2a	1·16977	−9·155	1·38901	11·889	45·727	83·492	0·637
A 3a	1·23727	−4·938	1·38882	4·631	29·527	57·216	0·665
A 4a	1·28061	−2·895	1·38894	2·164	20·649	40·787	0·683
A 5a	1·35302	−0·552	1·38877	0·330	6·511	13·894	0·713
A 6a	1·35803	−0·372	1·38885	0·348	5·807	12·430	0·715
A 7a	1·38207	−0·100	1·38883	0·055	2·120	5·143	0·724
A 8a	1·38245	−0·035	1·38881	0·115	2·092	5·091	0·724
A 8a•	1·26258	−13·431	1·38872	−0·032	1·006	3·350	0·680
A 9a	1·38563	−0·116	1·38889	−0·005	1·441	3·896	0·726
A10a	1·17273	−49·549	1·38897	−0·798	−2·755	−3·245	0·647
A11a	1·39125	−0·101	1·38887	−0·067	0·071	1·196	0·728
A12a	1·39168	−0·083	1·38893	−0·056	−0·062	0·918	0·728
A13a	1·39211	0·026	1·38883	0·036	−0·961	−0·977	0·728
A14a	1·39119	0·019	1·38877	0·040	−1·302	−1·662	0·728
A15a	1·38793	0·171	1·38888	0·248	−1·742	−2·745	0·726
A16a	1·38668	0·257	1·38885	0·344	−1·817	−2·988	0·726
A17a	1·38371	0·270	1·38885	0·504	−1·943	−3·394	0·724
A18a	1·37003	0·647	1·38872	1·461	−1·971	−4·381	0·719
A19a	1·28235	2·691	1·38882	11·883	5·212	−0·282	0·681

Table V. $a'_{0,6}$.

Colour	D			V	
Form	$h = 0$	$h = 20$	$h = 30$	$h = 0$	$h = 30$
A 1b	664·237	664·067	664·213	664·202	663·202
A 2b	673·896	674·020	673·916	673·885	673·064
A 2b•	694·126	693·978	694·136	694·146	694·114
A 4b	683·370	683·527	683·357	683·307	682·073
A 5b	690·721	691·017	690·729	690·687	688·332
A11b	692·860	694·582	692·866	692·797	687·404
A18b	691·712	702·500	691·639	691·685	677·171
A19b	685·861	791·588	685·931	685·799	623·054

Table VI. $f_{0,6}$.

Colour	D			V	
Form	$h = 0$	$h = 20$	$h = 30$	$h = 0$	$h = 30$
A 1b	720·009	721·053	723·223	721·213	723·570
A 2b	720·099	721·402	722·974	721·087	723·274
A 2b•	720·030	720·260	721·100	720·580	721·808
A 4b	719·927	721·923	724·150	720·624	723·797
A 5b	720·132	724·831	730·191	720·703	728·658
A11b	720·022	732·210	744·070	720·553	739·595
A18b	719·908	752·495	770·417	720·562	756·436
A19b	720·024	883·610	838·339	720·865	764·478

Table VII.

Form	$\kappa_0 \times 10^3$	A_0	$\kappa \times 10^3$	A	B	C	ϖ
A 1b	0·68484	− 1629·724	1·38887	1·377	− 2·379	− 0·653	0·437
A 2b	0·92317	− 292·109	1·38870	− 0·616	− 4·604	− 8·601	0·536
A 2b*	1·19565	− 57·360	1·38883	1·000	− 0·116	− 0·616	0·646
A 4b	1·10634	− 64·803	1·38903	− 0·963	− 7·450	− 13·818	0·612
A 5b	1·18423	− 33·189	1·38864	− 1·818	− 17·474	− 33·486	0·644
A11b	1·10685	− 106·681	1·38885	− 10·193	− 47·036	− 85·637	0·613
A18b	0·85555	− 975·209	1·38907	− 56·576	− 130·419	− 211·173	0·511
A19b	0·24847	− 211286·449	1·38884	− 380·455	− 542·531	− 718·636	0·264

Table VIII. $a'_{0,6}$.

Colour	D			V	
Form	$h = 0$	$h = 20$	$h = 30$	$h = 0$	$h = 30$
B 1a	686·105	683·052	686·101	686·072	689·313
B 2a	694·402	693·570	694·420	694·400	696·279
B 3a	703·377	703·291	703·410	703·344	704·257
B 4a	710·359	710·358	710·336	710·331	710·693
B 5a	712·936	712·957	712·976	712·884	713·306
B 6a*	743·600	743·684	743·651	743·595	743·661
B 6a	713·727	713·720	713·713	713·632	713·981
B 7a	715·634	715·622	715·639	715·618	715·983
B 8a	721·907	721·835	721·913	721·886	722·509
B 9a	723·535	723·436	723·534	723·521	724·302
B10a	734·872	735·077	734·823	734·863	733·231
B11a	732·139	732·750	732·108	732·083	729·343

Table IX. $f_{0,6}$.

Colour	D			V	
Form	$h = 0$	$h = 20$	$h = 30$	$h = 0$	$h = 30$
B 1a	720·000	715·020	716·216	720·609	720·308
B 2a	719·940	717·839	717·253	720·384	719·705
B 3a	720·171	719·475	718·851	720·379	720·026
B 4a	719·943	719·947	719·931	719·987	720·427
B 5a	720·051	720·338	720·693	720·012	721·104
B 6a*	720·013	720·055	719·934	718·665	718·574
B 6a	719·940	720·302	720·760	719·838	721·089
B 7a	720·008	720·610	721·400	719·943	721·768
B 8a	720·024	721·709	724·011	719·798	724·519
B 9a	720·061	722·151	725·024	719·802	725·683
B10a	719·987	717·216	713·106	719·435	710·725
B11a	720·036	714·809	706·513	719·479	702·864

Table X.

Form	$\kappa_0 \times 10^3$	A_0	$\kappa \times 10^3$	A	B	C	ϖ
B 1a	1·15945	−4·357	1·38889	17·270	24·706	33·557	0·634
B 2a	1·27469	−0·684	1·38900	5·003	9·726	15·640	0·681
B 3a	1·35274	−0·313	1·38856	0·575	2·792	6·057	0·713
B 4a	1·38538	−0·142	1·38900	−0·017	−0·036	0·931	0·726
B 5a	1·39013	−0·126	1·38879	−0·080	−1·039	−1·019	0·728
B 6a*	1·22893	−18·375	1·38886	−0·456	−0·350	1·182	0·660
B 6a	1·39143	0·006	1·38901	0·030	−1·298	−1·649	0·728
B 7a	1·39236	0·086	1·38887	0·078	−2·124	−3·346	0·728
B 8a	1·38121	0·115	1·38884	0·459	−5·830	−11·050	0·723
B 9a	1·37505	−0·024	1·38877	0·620	−7·212	−13·922	0·721
B10a	1·27655	−0·012	1·38891	−1·326	9·207	19·974	0·679
B11a	1·26302	−8·438	1·38882	−3·788	16·760	37·168	0·673

Table XI. $a'_{0,6}$.

Colour	*D*			*V*	
Form	$h = 0$	$h = 20$	$h = 30$	$h = 0$	$h = 30$
B 1b	788·736	686·978	788·685	788·647	845·693
B 2b	760·641	736·938	760·611	760·617	781·710
B 3b	745·667	739·785	745·690	745·629	754·883
B 5b	735·068	733·979	735·090	735·076	738·925
B11b	741·788	741·929	741·790	741·799	740·987

Table XII. $f_{0,6}$.

Colour	*D*			*V*	
Form	$h = 0$	$h = 20$	$h = 30$	$h = 0$	$h = 30$
B 1b	719·968	658·580	811·903	717·534	879·090
B 2b	720·067	718·623	772·382	718·611	793·620
B 3b	720·065	727·894	751·889	719·087	761·049
B 5b	720·033	726·515	737·356	719·445	741·071
B11b	719·968	718·853	717·049	719·290	715·381

Table XIII.

Form	$\kappa_0 \times 10^3$	A_0	$\kappa \times 10^3$	A	B	C	ϖ
B 1b	0·30474	−30490·898	1·38895	513·623	354·274	221·698	0·269
B 2b	0·84110	−468·566	1·38876	129·024	48·909	−20·984	0·495
B 3b	1·11583	−62·965	1·38876	34·886	−14·287	−58·087	0·611
B 5b	1·27888	−9·387	1·38883	6·695	−20·261	−44·874	0·680
B11b	1·21815	−17·790	1·38895	−0·848	3·553	8·329	0·654

CURVATURE MEASUREMENTS

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Read and discussed, 15th December, 1920.

Abstract. The paper contains a description of the arrangements of well-known methods of spherometry which the author has found most suitable for accurate determinations, and also of a new microscope method suitable for medium and large curvatures.

In a previous paper to the Society the writer described a new type of spherometer* which materially increases the ease and accuracy of curvature measurements for lenses at least 1 cm. in diameter, provided the curves are not too shallow. There are cases, however, for which a spherometer is inapplicable. For very small lenses, such as may be encountered in microscope objectives, or for lenses of very shallow curvature, other methods have to be adopted. Other circumstances, e.g. smallness of aperture, may prevent the use of a spherometer even for medium curvatures, and it is necessary in a testing laboratory to have alternative methods to cover the whole range which may be encountered. There are a great variety of such methods, but they are not all of satisfactory accuracy or convenience to be of practical value. In this paper I propose to describe the particular variants of one or two well-known methods which I have found to be convenient in practice, and also a new magnification method which is suitable over a very wide range of curvature.

Newton's Rings. Of the old methods the most generally applicable is that which makes use of Newton's rings. It is the only satisfactory method for convex curves of long radius, but its utility is by no means limited to such cases, as, by employing suitable arrangements, it can be used down to quite small radii. It is not necessary here to explain the theory of the method, which is well known. We need only concern ourselves with the experimental arrangements. For shallow curves a good quality travelling microscope, such, for instance, as those employed for the measurement of spectrograms, is essential. The lens is laid on the stage of the microscope, Fig. 1, with the surface to be tested upwards, and the flat plate laid on top of it. Unsteadiness can be prevented by little pellets of plasticene or soft wax near the periphery. The illumination should be supplied in a parallel beam at exactly normal incidence by means of an arrangement of the type shown in the diagram. *A* is a small aperture behind which is placed a strong source of monochromatic light. The aperture is at the focus of a lens *L*. The light is reflected by a transparent glass reflector, *R*, which is mounted in a bracket with universal adjustment, attached to the framework which carries the microscope.

* J. Guild, *Trans. Opt. Soc.* 19 (1918), 103.

In making the actual measurements it is necessary to measure the diameters of two rings, the n_1th and the n_2th. The accuracy of the final result will be greater the greater $n_2 - n_1$. It is not desirable, however, that *either* n_2 or n_1 should be small, as the rings near the centre being broad and diffuse the accuracy of setting is considerably less than for rings further out. The effect on the result of errors in setting is readily deduced by differentiating the equation

$$r = \frac{D_{n_2}^2 - D_{n_1}^2}{4(n_2 - n_1)\lambda},$$

which connects r, the radius of curvature, with D_{n_1} and D_{n_2} the diameters of the n_1th and n_2th rings, with respect to D_{n_1} and D_{n_2} separately. It requires D_{n_2}/D_{n_1} times as big an error in D_{n_1} as it requires in D_{n_2} to produce a given error in r. Thus the relative accuracy with which the diameters have to be measured is proportional to the diameters. It is easy to show that the widths of the fringes are inversely

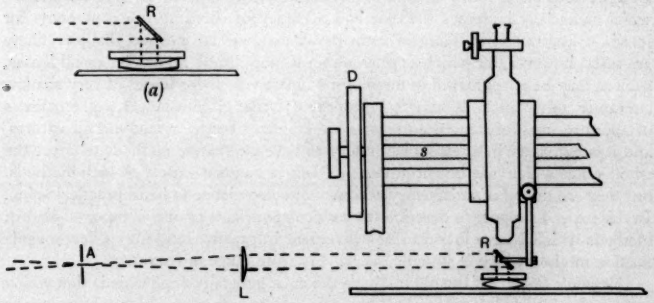

Fig. 1.

proportional to their diameters, so that if settings were accurate to the same fraction of a fringe width, the probable error in the result would be of the same order in each measurement. As a matter of fact the accuracy of setting does vary very nearly in this way at some distance from the centre, but for the first few fringes the fringe width is varying rapidly and the judgment of the setting is affected by the want of symmetry. It is not desirable therefore to make measurements on rings of smaller order than the 6th or 7th. For the outer ring as large a value of n as possible should be taken. This involves working where the fringes are very close together, and it will be found of convenience to employ a micrometer eyepiece with the microscope. The microscope as a whole is moved slowly across the fringe system by means of the screw s, the fringes being counted as they pass the crosslines. When the fringe is reached at which it is intended to stop, the reading on the drum D is taken. Several settings are then made by means of the micrometer eyepiece, and the mean of these readings used to correct the drum reading. The advantage

of this process is that if the repeated settings are made with the screw s, the cross-lines have to be brought back several fringes each time in order to take up the backlash of the screw. It is very easy in doing this to get confused as to the fringe on which the measurement is being made. If the repetitions are made with the micrometer eyepiece, however, the screw being of finer pitch and the mechanism being more delicate altogether it is not usually necessary to retreat more than a fringe width to take up the backlash, and the chances of getting on the wrong fringe are minimised.

There are one or two details in the arrangement of Fig. 1 which differ from those usually described. In the descriptions hitherto given the lens has always been placed on the flat surface, and in consequence the fringes are seen *through* the lens, and are in general magnified. An object in contact with one surface of a lens is magnified, when observed through the lens, in the ratio f/f_b where f is the focal length and f_b is the "back focal length," i.e., the distance from the principal focus to the surface of the lens. It is only in the special case of a lens in which the other surface is flat, and in which, therefore, the appropriate nodal point coincides with the curved surface, that $f_b = f$. In all other cases the distance traversed by the microscope has to be multiplied by f_b/f to give the true diameters of the rings. In the diagrams illustrating the method this point is always avoided by the convenient choice of a plano-convex lens; but in practice this is the exceptional case. The expedient here adopted of placing the flat uppermost as in Fig. 1 overcomes this difficulty entirely.

It is important to ensure illumination by a parallel beam at normal incidence for all positions of the microscope. With the ordinary method of illumination, in which the light comes as a converging cone from the objective, fringes are only visible with extremely thin air films, and consequently the lens and flat must be in contact. This being so the expedient is adopted, in dealing with concave surfaces, of measuring the difference in curvature between the surface and a convex surface of greater curvature which rests within it. But this is an inaccurate arrangement except when the two curvatures are very nearly equal, as an error in the assumed value of the radius of the convex surface produces a greater percentage error in the result obtained for the concave.

Suppose h is the vertical distance between the two surfaces at a distance ρ from the centre. Let h_1 and h_2 be the respective distances of the convex and concave surfaces above a plane tangential to them at their point of contact. Then if r_1 is the radius of the convex and r_2 that of the concave surface

$$\frac{1}{r_1} = \frac{2h_1}{\rho^2}; \quad \frac{1}{r_2} = \frac{2h_2}{\rho^2},$$

$$\therefore \frac{1}{r_1} - \frac{1}{r_2} = \frac{2}{\rho^2}(h_1 - h_2).$$

To find the effect of an error in r_1 differentiate this, regarding the right-hand side as constant, then

$$\frac{1}{r_1^2}dr_1 = \frac{1}{r_2^2}dr_2 \quad \text{or} \quad \frac{dr_1}{r_1} \times \frac{r_2}{r_1} = \frac{dr_2}{r_2}.$$

Thus a proportionate error dr_1/r_1 in the assumed value of r_1 gives rise to a proportionate error dr_2/r_2 which is r_2/r_1 times as great.

With a beam of parallel light at normal incidence, as shown in Fig. 1, it is possible to get and measure fringes with a large total path difference, and so to measure the radius of a concave surface directly against a flat, which is laid on the rim of the lens, as in Fig. 1 (*a*). In order to obtain visible fringes with thick films it is essential that the angular diameter of the pencil of rays employed for each part of the fringe system should be small, otherwise variations of phase relationship exist within the pencil due to the different obliquity with which the rays meet the film, and the fringes are obliterated. For a given width of pencil the variation of phase is least at normal incidence. Hence the need for collimating the incident beam, for if the source were not at infinity, the variation of retardation due to varying angle of incidence from the centre outwards would vitiate the result. In making the adjustments remove L and adjust R by hand until the fringes appear as distinct as possible. Replace L, which should be mounted on a stand having rack and pinion adjustments both transversely and vertically, and adjust it until the rings are equally distinct in all parts of the field. A guide to the correct adjustment will be found in the size of the rings near the centre. Since for normal incidence the retardation is a maximum, the rings shrink to a minimum size when the incidence is normal. The movements of the rings with slight changes in the adjustment of the light give rise to the impression that the results obtained depend entirely on the adjustment. This is not so, however, as the *relative* positions of different parts of the ring system are affected by the adjustment of the light to exactly the same extent as if the surfaces were in contact as in the ordinary method.

The aperture A should be as small as possible and this necessitates the use of a very bright source of light. The green line of mercury is the most useful to employ and should be obtained from a quartz arc made specially for interference work*. The other lines in the spectrum are much less intense than the green line, and only detract slightly from the blackness of the fringes, the positions of which are determined by the green line. The best type of aperture to use at A is a small iris diaphragm which can be adjusted to give the best compromise between brightness and distinctness.

In employing Newton's rings for medium and steep curves, the travelling microscope ceases to be satisfactory on account of the smallness of the radii of the rings. It is better in this case to use an ordinary microscope with a micrometer scale in the focal plane of a Ramsden eyepiece, and to measure the rings on this scale. The objective must be of such power as to give an image of the ring system of suitable size. With steep curves it requires to be of too high a power to allow the interposition of the vertical illuminator shown in Fig. 1, and an ordinary vertical illuminator above the objective must be employed. The arrangement cannot then be used for concave surfaces. However, for medium and steep curves the method is no more accurate and is much more tedious than several other methods which will now be described.

* J. Guild, *Proc. Phys. Soc.* 32 (1920), 341.

Microscope Methods. The following methods require a good microscope stand with mechanical stage and a universal substage with centering adjustments. A vertical illuminator of the ordinary inclined cover glass type is mounted above the objective.

(*a*) *For Concave Surfaces.* The lens is mounted on a support attached to the substage fitting, which can be raised or lowered by the rack and pinion, *P*, Fig. 2 (*a*), and centered by the screws *s*, *s*. A piece of thin cover glass on which some easily

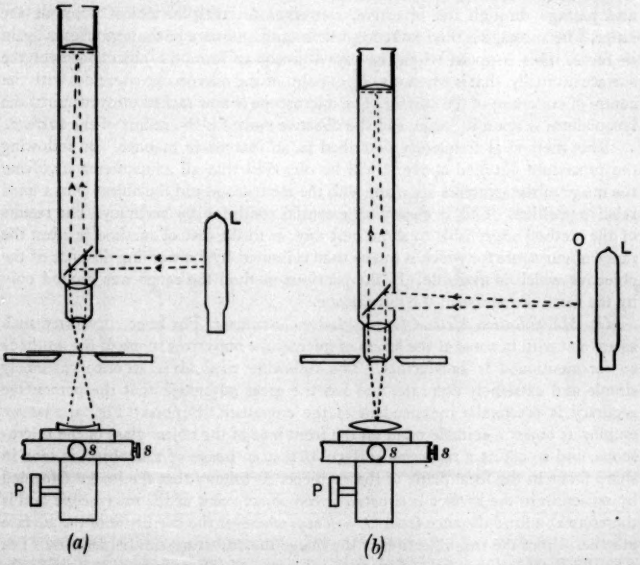

Fig. 2.

recognisable markings have been scratched is placed on the stage, and the microscope is focussed on these, vertical illumination being supplied by the lamp *L*. The substage is then racked up or down until the image of the markings formed by reflection at the concave surface is also seen in sharp focus. By means of the screws *s*, *s* the image is centered with respect to the object, which is then at the centre of curvature of the surface. The cover glass is removed and the microscope racked down until some lycopodium grains sprinkled on the surface of the lens are in focus. The distance the microscope has been moved is the radius of curvature required. The method is accurate to about 0·05 mm.

(*b*) *For Convex Surfaces.* For this case the arrangement is somewhat more complicated. The microscope is initially racked down so that the objective is about the level of the stage. The substage is adjusted until some lycopodium grains on the surface of the lens are in focus. The surface is then at the object point* of the microscope. A piece of silvered glass with a number of scratches cut on it is placed at O, Fig. 2 (*b*), and its distance from the vertical illuminator is adjusted until an image of the scratches is seen simultaneously with the lycopodium. Under these circumstances light from the scratches, after reflection at the vertical illuminator and passage through the objective, converges towards the object point of the latter. The substage is then racked upwards until an image of the scratches is again in focus. This happens when the rays converging from the objective meet the surface normally, that is when the object point of the microscope coincides with the centre of curvature of the surface. The microscope is now racked upwards until the lycopodium is again in focus, and the distance moved is the radius of the surface.

This method is frequently described in an inaccurate manner. By following the procedure outlined above it will be observed that all adjustments involving the image of the scratches are made with the microscope and the object O in a fixed relative position. This is clearly an essential condition for accuracy. The results of the method are reliable to about 0·05 mm. as in the case of method (*a*); but the range of curvature for which it can be used is limited by the working distance of the objective which is available. In the previous method the range was limited only by the vertical traverse of the microscope.

(*c*) *Magnification Method for very Large Curvatures.* For large curvatures such as are met with in some of the lenses of microscope objectives, none of the methods so far mentioned is satisfactory. The following method is at once extremely simple and extremely accurate, and has the great advantage that the percentage accuracy is practically independent of the curvature. Suppose, Fig. 3, that we employ as object a suitable mark on the front lens of the object glass of the microscope, and so adjust a reflecting surface M that an image of this object is seen in sharp focus in the focal plane of the eyepiece. It follows that the image I formed by reflection in the surface is situated at the object point of the microscope and is therefore at a fixed distance from O, $= k$ say, whatever the curvature of the surface may be. From the magnification of the image the curvature can be deduced. Let u be the distance from the surface to the object, v the distance from the surface to the image and r the radius of the surface. Then $\frac{1}{u}+\frac{1}{v}=\frac{2}{r}$, quantities being regarded as positive when measured against the incident light. $k = IO = u - v$. The magnification of the image at $I = -v/u = m$ say.

Hence $$1 - m = 2v/r; \quad 1 - 1/m = 2u/r.$$

$$\therefore\ m - 1/m = 2k/r \qquad \text{.....(i)}$$

or $$r = \frac{-2mk}{1-m^2} \qquad \text{.....(ii).}$$

* For convenience of description we may use this term to denote the point conjugate to the centre of the graticule in the focal plane of the eyepiece.

From (i) we see that in general there are two distances of the mirror from the object which satisfy the condition of the experiment, namely the value of u which gives a magnification m, and another value u' for which the magnification is m', where $m' = -1/m$. In the case of a convex surface one of these values is negative and so is experimentally inadmissible. The other value is such that the magnification is positive and numerically less than unity. For a concave surface, both values of u are positive. In one case u is less than $r/2$ and the magnification is positive and greater than unity. In the other case u is greater than r, and the magnification is negative and numerically less than unity.

This method can be employed over a large range of curvature; but the best experimental conditions differ for different ranges. For very short radii the

Fig. 3.

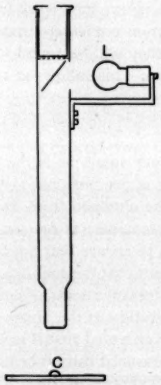

Fig. 4.

arrangement of Fig. 4 is convenient. The microscope is fitted with a vertical illuminator, preferably of the type shown in which the reflecting coverslip is just under the eyepiece. Light is supplied by a small 4 volt lamp, attached for convenience to the microscope. The illuminated aperture of the lens itself serves as the object. The surface, C, to be measured is placed centrally on the stage, and on racking the microscope up or down a circular disc will be seen which is the image by reflection in C of the aperture of the objective. This is sharply focussed and its diameter measured on a micrometer scale in the focal plane of the eyepiece. The actual dimensions depend, of course, on the magnification of the microscope as well as on the magnification of the reflected image. However, for a given objective and tube length the measured diameter is proportional to the magnification m. The instrument is calibrated by measurements on a series of the steel spheres sold for

ball bearings. These can be obtained of diameters ranging from 1/32nd of an inch to 3 inches by small steps. The sizes up to about an inch make excellent optical standards, and a complete set should be available in any laboratory. They can be measured with a micrometer gauge; but it will be found that they are invariably so close to their nominal values that for optical purposes these values can be taken as substantially correct.

The accuracy with which the diameter of the luminous disc can be measured depends, of course, on its size. An objective should be employed which gives an image of at least 4 mm. on the eyepiece scale. The size of the image depends not only on the power of the objective but also on its working distance and the linear dimensions of the aperture. However, it is easy to find a series of three or four objectives which between them will cover the range from the smallest radii encountered in microscope work up to radii of about 10 mm. If, by using the steel balls, calibration curves are drawn for each objective for the range in which it is to be used, they will be found to be practically straight lines within the accuracy of observation. This follows at once from the relation

$$r = -\frac{2mk}{1 - m^2}$$

when m is very small, as is the case with surfaces of such short radii. Thus the scale reading is proportional to the radius. By a single reading the radius of any surface can be obtained from the calibration curve. It is safer, however, to make check measurements on one or two of the spheres each time the apparatus is reassembled, to ensure that the constant of the instrument has not been altered.

It is evident that the calibration curve obtained with the spheres will do equally well for the measurements of concave curves, the real image formed being of the same magnification as the image obtained with a convex curve of the same radius.

It will be observed that if suitable objectives are employed, the diameter of the disc to be measured can be kept of the same order of magnitude (40 to 100 scale divisions) however small the radii tested may be. Since it is easy to read to ·1 scale division (with a good quality micrometer scale) the average sensitivity of the method is about 1 part in 700. Further, the method is virtually one of substitution, and all aberrational effects of the observing microscope are eliminated. Thus there is no source of error other than the accidental errors of measurement, and the accuracy is equal to the sensitivity.

In practice the definition is improved by using a light green filter to cut out the extremities of the spectrum, as, if white light is used, chromatic aberration tends to spoil the sharpness of the image.

Magnification Method for Medium Curvatures. It is not convenient to use the method in the form described above for radii much over 1 cm., because the disc becomes too large to be contained in the field of view with the lowest power objectives. For larger radii it is better to use as object a scale, similar to that employed in the eyepiece, attached to the front of the objective with soft wax or plasticene. An image of this scale is seen when the object is at the correct distance

from the surface. By measuring the dimensions of a suitable number of graduations, the magnification can be obtained from a knowledge of the magnification of the microscope. We may, if we choose, determine k, and calculate the radius directly from formula (ii) above. To measure k a flat surface, preferably silvered, is laid on the stage and the microscope focussed on some scratches on the silver. On racking downwards, an image of the scale comes into focus. The distance which the microscope has been moved is clearly $\frac{1}{2}k$.

As in the case of very small radii, however, it is better to calibrate the instrument empirically. Unfortunately there are no easily available standards such as the steel balls employed in the former case; but points on the curve can be obtained from measurements on lens surfaces which have been determined accurately with a spherometer. If sufficient of such standards are available to determine the calibration curve accurately, an unknown radius can be obtained directly from the curve; but it is usually better to measure m for the surface in question and also the corresponding value for the *nearest* standard, and deduce the unknown radius from the formula

$$r = r_1 \frac{m\,(1 - m_1^2)}{m_1\,(1 - m^2)}.$$

In the case of concave curves, if the radius does not exceed two or three centimetres it is easy to obtain the real and diminished image, and in this case the convex standards apply to concave curves as well. With shallower curves, however, the real image is too far from the lens to be within the range of vertical traverse of ordinary microscopes; and it is necessary to work with the virtual image, for which, as we saw earlier, m is greater than unity. A separate set of concave standards is necessary for these. Unfortunately as the magnification increases the image of the scale becomes very faint and the measurement is somewhat difficult with magnifications greater than unity. The range of curvature for which the diminished real image can be utilised may be increased by mounting the lens on the substage or even on a separate support below the microscope.

Except with comparatively small radii the 4-volt lamp L (Fig. 4) will not give a sufficiently bright field. It is better to employ a 100 c.p. "Pointolite" lamp, or a similarly intense source, behind a ground glass screen situated about 30 cm. from the microscope and on a level with the aperture of the illuminator. Except for the ground glass the lamp should be entirely enclosed.

The scale employed as object should not be too fine. A 1½ inch objective with a working distance of about an inch will give satisfactory results over most of the range for which the method is applicable.

There are a variety of other magnification methods; but they involve the measurement of two magnifications and are inferior in convenience and accuracy to that just described.

Direct Location of Centre. For concave curves of radii above 20 cm. this method may be employed provided the ratio of the diameter of the surface to the radius of curvature is not too small. The principle of the method is that if an object and its image by reflection in the surface are coincident, or in close juxtaposition in

a plane perpendicular to the axis of the surface, they are located at the centre of curvature. The method as usually described, in which the image is formed on a card, is not, however, very sensitive; and if the best possible accuracy is required in the result the following arrangement will be found much better. A microscope slip is silvered or platinised on one side, and a network of very fine scratches is made with a sharp needle. The slip is mounted on the stage of a microscope which is turned back so that the microscope tube is horizontal. The scratches are illuminated by means of a small prism *P*, Fig. 5, and a "Pointolite" lamp or other small concentrated source at *S*. The concave surface *M* is arranged so that an image is formed on the slip a little to the side of the illuminating prism *P*. In this region the plating should be almost all scraped off, a few streaks being left on which to focus the microscope. When the adjustment is correct, these streaks and the reflected image of the illuminated scratches will be simultaneously in focus. The plated side of the slip should be towards the mirror, and the distance between them can be

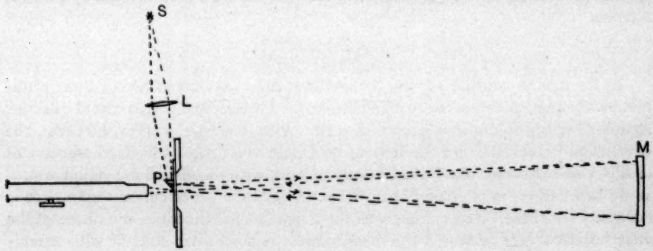

Fig. 5.

measured in any convenient manner. It is advantageous in practice not to attempt precise adjustment of the mirror, but to get it approximately correct and measure the error by racking the microscope so that first the surface of the slip is in focus and then the reflected image. Half the distance between the two positions is the correction to be applied to the distance from slip to mirror. The edges of a silvered film can be focussed up with a microscope more critically than almost any other type of object, and the depth of focus is reduced to a minimum. The accuracy depends on the ratio of diameter to radius of curvature. It is about 0·1 millimetre for a ratio of 1 : 20.

The direct location of the centre is unfortunately impossible with convex surfaces. The method in which an auxiliary lens is used to produce a convergent beam, which meets the convex surface normally and is reflected back through the lens and focussed in the vicinity of the object, is valueless for work of any accuracy on account of the aberrations of the lens.

The methods here described, in conjunction with a good spherometer, have been found to provide adequately for curvatures of all magnitudes. Where to leave

off one method and to use another depends largely on the predilections of the observer, as their ranges of utility overlap. I have intentionally omitted reference to methods—some of them very useful for works practice—in which the possession of both the concave and convex curves of any radius is assumed, since the circumstances in which these can be used rarely arise in a general testing laboratory.

DISCUSSION

Prof. A. F. C. Pollard (communicated remarks): The measurement of the curvature of lenses raises a point which some perhaps may consider a little premature when viewed in the light of present day lens manufacture. Nevertheless the matter may be of interest to the Society.

It would appear that all methods of measuring the radius of curvature of curved surfaces measure the curvature over a comparatively large part of the surface which is assumed to be spherical. I do not know of any instrument or method which measures the radius at a point on the surface. It is obviously impossible to do this at a mathematical point on the surface, but it is possible in the case of positive curvature to reduce the part of the surface about the point over which the radius is measured to a very small fraction of the whole. By the following method this part of the surface is reduced to such a short line that the curvature is determined practically at a point of the surface. If a plane touch a curved surface and be rolled through some small geodesic arc of length s and in doing so rotates through an angle θ, the radius ρ over the arc of rolling is given by $\rho\theta = s$. To determine ρ it is only necessary to measure s and θ.

s can be measured to an accuracy of at least $\pm$ ·001 mm. if the Newtonian ring system at the point of contact be observed, through a rolling glass plate, in the micrometer eyepiece of a microscope. But the measurement of θ is not so easy. The mirror and scale method is clumsy and introduces a number of possible errors. It occurred to me that θ could be measured in the following way, which introduces only two errors, one of which is of the same order as that due to s. Let the plane touch the surface to be tested of radius ρ_1 and also a standard sphere of radius ρ_2. If now the plane roll through a small angle the points of contact on the surfaces ρ_1 and ρ_2 describe small arcs s_1 and s_2 respectively, which in general are not geodesic. Then since the plane generates the enveloping cone to the two spheres ρ_1 and ρ_2,

$$\rho_1/\rho_2 = s_1/s_2,$$

and when $\rho_1 = \rho_2$ the arcs become geodesic lines and the cone a cylinder. So that to compute ρ_1 it is only necessary to measure the absolute dimension of ρ_2 the standard sphere, which is one of the simplest and most accurate of metrological processes. Since s_1 and s_2 enter the equation as a ratio, any unit may be used to measure these lengths.

This principle of reducing the elemental absolute measurements to a minimum is as important in instrument design as that of determining measurements from sums or differences of readings taken by the same physical procedure and both are frequently neglected.

The worst possible error $\delta\rho_1$ of ρ_1 is given by

$$\frac{\delta\rho_1}{\rho_1} = \pm\left(\frac{\delta s_1}{s_1} + \frac{\delta s_2}{s_2} + \frac{\delta\rho_2}{\rho_2}\right).$$

Now the errors δs_1 and δs_2 may each not exceed $\pm$ ·001 mm. and the error $\delta\rho_2 = \pm$ ·0025 mm., if the steel spheres manufactured by the Hoffman Manufacturing Co. be used as standards. If we take $\rho_2 = 10$ mm., $s_2 = 1$ mm. say, the worst possible error ranges from $\pm$·011 mm. on a 1 mm. radius surface to $\pm$ ·023 mm. on a 10 mm. radius surface. The length of the arc on the 1 mm. radius surface is 0·1 mm. and the radius is measured over this arc to $\pm$ ·011 mm.

As the plane rolls over the two surfaces any point of it will describe a tortuous curve, so that if the two points of contact serve as two points of support to a horizontal plate the third point must be supported so as to have free motion in a horizontal plane. The simplest way to do this is to float the third point of support upon mercury and vary the level of the mercury in order to roll the plate upon the two surfaces. In this way it would be possible to map out the positive curvature over the surface of an aspherical lens, say, and minute lenses could be measured at once.

The design of an instrument, based upon the above notions, which can be used as a lens comparator or as a spherometer is now engaging my attention.

NOTE ON PYRAMIDAL ERROR IN PRISMS

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Abstract. The effect of pyramidal error on the sum of the three angles of a triangular prism is discussed and a convenient method of measuring pyramidal error on an ordinary spectrometer is described.

It is frequently necessary both in goniometry and spectrometry to assume a value for the sum of the three angles of a triangular prism, which is usually taken as 180°. In work of precision it is necessary to bear in mind that this is only strictly true if the prism is a true prism with its three edges parallel. The angles of a prism as determined by the usual methods of goniometry are the actual angles between the faces and these are only equivalent to the angles of a plane triangle if the normals to the faces are coplanar. If the normals are not coplanar, i.e., if the three edges of the prism are not parallel, the prism is really part of an elongated pyramid.

There is no universally agreed method of defining pyramidal error quantitatively; we will define it here as the angle between one of the edges and the opposite face. In the case of an equilateral prism the value is the same whichever edge is considered, but in other cases it is not so. For any isosceles prism it is convenient to take as the pyramidal error the angle between the unique face and the opposite edge.

Let *OBC*, Fig. 1, be the plane of the face in question and *OA* the opposite edge. Let *ABC* be a section of the prism by a plane perpendicular to *OA*. Draw *AP* perpendicular to *BC*. Then $A\hat{O}P$ is the pyramidal error p, as above defined. Denote the angle between the planes *AOB* and *AOC* by $\hat{A}$ and the other interplanar angles by $\hat{B}$ and $\hat{C}$. $B\hat{A}C = \hat{A}$, but $\hat{B} > A\hat{B}C$, and $\hat{C} > A\hat{C}B$. Thus

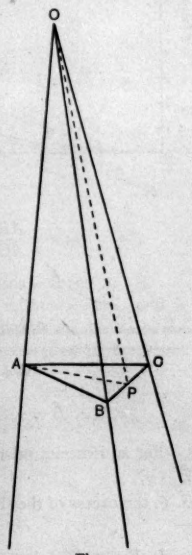

Fig. 1.

$$\hat{A} + \hat{B} + \hat{C} > B\hat{A}C + A\hat{B}C + A\hat{C}B.$$

These are the angles of a plane triangle and so their sum = 180°.

$$\therefore \hat{A} + \hat{B} + \hat{C} > 180°.$$

Denote the angles $O\hat{B}A$, $O\hat{B}C$ and $A\hat{B}C$ by α, β, and γ respectively; then from the geometry of the solid angle of which α, β, γ, and $\hat{B}$ are elements,

$$\cos\gamma = \cos\alpha\cos\beta + \sin\alpha\sin\beta\cos\hat{B},$$
$$\therefore \cos\hat{B} = \cos\gamma\,\mathrm{cosec}\,\alpha\,\mathrm{cosec}\,\beta - \cot\alpha\cot\beta$$
$$= \cos\gamma\,(1 + \tfrac{1}{2}\cot^2\alpha + \tfrac{1}{2}\cot^2\beta) - \cot\alpha\cot\beta,$$

since α and β are nearly right angles.

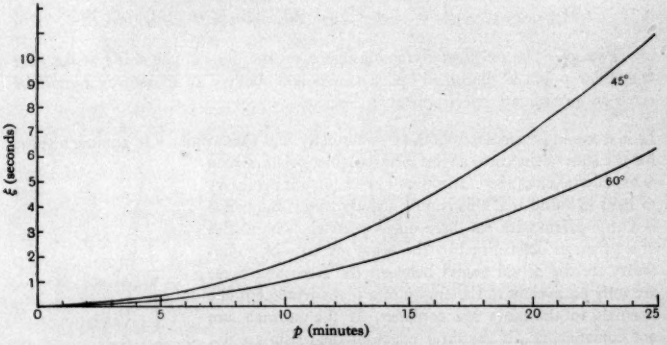

Fig. 2.

$$\cot\alpha = \frac{AB}{AO} = \frac{AB}{AP}\cdot\frac{AP}{AO} = \frac{p}{\sin\gamma};\ \text{similarly}\ \cot\beta = \frac{p}{\tan\gamma},$$
$$\therefore \cos\hat{B} = \cos\gamma\left\{1 + \frac{p^2}{2}\left(\frac{1}{\sin^2\gamma} + \frac{1}{\tan^2\gamma}\right)\right\} - \frac{p^2}{\sin\gamma\tan\gamma}$$
$$= \cos\gamma - \frac{p^2}{2}\cos\gamma$$
$$= \cos\left(\gamma + \frac{p^2}{2}\cot\gamma\right),$$
$$\therefore \hat{B} = \gamma + \frac{p^2}{2}\cot\gamma.$$

For an isosceles prism, it is clear from symmetry that $\hat{C} - A\hat{C}B$ is equal to $\hat{B} - \gamma$,

$\therefore \xi$, the excess of the three prism angles over 180°,

$$= 2(\hat{B} - \gamma) = p^2\cot\gamma.$$

In Fig. 2 the values of ξ for various pyramidal errors are plotted for an equilateral prism and a 45°, 45°, 90° prism, the two commonest types. Evidently the sum of the angles may be taken as 180° without appreciable error if p does not exceed a few minutes. For larger values ξ increases rapidly and must be taken into account in precise measurements.

Measurement of p.

We may conveniently measure p on an ordinary spectrometer, of which the slit has been turned into a horizontal position, in the following manner. Let ABC, Fig. 3, be the prism, arranged as shown with respect to the collimator. Rays such as $S'U$ are reflected from the face AB in the direction UW'. Rays such as ST follow the path $STUVW$, VW being parallel to UW' if $\hat{A} = \hat{B}$ and $p = 0$. If $\hat{A}$ is not exactly equal to $\hat{B}$ there will be a slight horizontal displacement; this is immaterial. If p is not zero there will be a slight vertical displacement. Take as reference plane a plane perpendicular to the edge C and let the incident light be in this plane. The normal NU will have an inclination p to this plane. Suppose it slopes downwards towards the left; then the reflected ray UW' slopes downwards by an angle $\phi = 2p \cos i$. Since AC is perpendicular to the reference plane, TU is also in this plane, so the ray UV slopes upwards by an angle $\phi' = 2p \cos i'$.

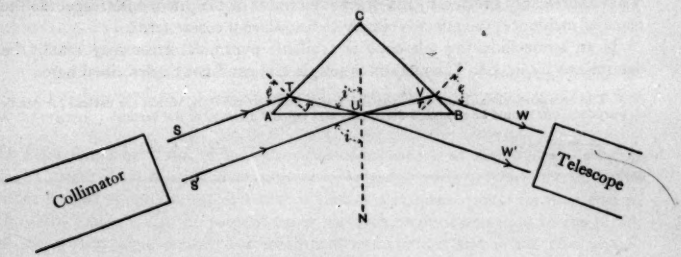

Fig. 3.

This ray meets BC, which is also perpendicular to the reference plane, and is refracted from glass to air. It is easily shown that ψ, the upwards inclination of the emergent ray VW, is equal to $\mu\phi'$, where μ is the refractive index of the prism. Thus the vertical displacement, δ, of VW with respect to UW'

$$= \phi + \psi = 2p\,(\cos i + \mu \cos i').$$

We must substitute for i' in terms of the angles of the prism and the angle of incidence i.

$$i' = \hat{A} + r = \hat{A} + \sin^{-1}\frac{1}{\mu}\sin e$$

$$= \hat{A} + \sin^{-1}\frac{1}{\mu}\sin\{\pi - (\hat{A} + i)\},$$

$$\therefore\ \delta = 2p\left[\cos i + \mu\cos\left\{\hat{A} + \sin^{-1}\frac{1}{\mu}\sin(\pi - \hat{A} - i)\right\}\right],$$

whence p is obtained.

To measure δ the simplest method is to employ a micrometer eyepiece rotated so that the traverse is vertical. If there is no micrometer eyepiece the vertical

distance of any point in the field of view of the telescope from the intersection of the crosslines can easily be determined by moving the telescope horizontally until first one of the crosslines* and then the other passes through the point. If x is the angle through which the telescope has been moved between the two settings and θ is the angle between the crosswires, then y, the distance which the intersection of the crosswires passes above or below the point in question, is equal to $x/2 \tan \theta/2$. By setting in this way on each image of the centre of the slit (which must be marked by a wire stretched across it) their vertical displacement can be measured.

In some instruments the provision for the vertical adjustment of the telescope may lend itself to the measurement of small vertical displacements. Whichever method of measuring δ is employed it is advantageous to block out first the rays externally reflected and then those internally reflected. Otherwise the overlapping of the images may be troublesome.

The advantage of the above method of determining pyramidal error is that δ is not appreciably affected by incorrect adjustment of the prism with respect to the plane of incidence; the effect is simply to introduce a cosine term.

If an autocollimating telescope is available pyramidal error may readily be determined by its aid. The method is simple and need not be described here.

* The crosslines should not be of the horizontal and vertical type, which are useless for nearly all purposes, but should be mounted diagonally and equally inclined to the vertical

ON CRITICAL ANGLE REFRACTOMETRY

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MS. received, 26th November, 1920. *Read, 15th December*, 1920.

Abstract. In this paper the authors discuss the possibilities, for work of high precision, of the critical angle method of measuring the refractive index of a prism. For general work the method is inferior in convenience and accuracy to the method of minimum deviation, but by a suitable arrangement of the apparatus it can be used to determine refractive indices near 1·53 without the use of a spectrometer, and serves as a useful check on more general methods which depend on the accuracy of a spectrometer circle.

In the second part of the paper the errors of setting in critical angle refractometry are discussed, and a simple method of determining the necessary correction is described.

In a previous paper* one of us dealt with various aspects of critical angle refractometry, particularly as applied to commercial refractometers in which the refractive index sought is determined in terms of that of a standard prism forming part of the refractometer. In the present paper we wish to draw attention to the utility of the critical angle method for absolute determinations and to describe experimental arrangements suitable for obtaining the maximum accuracy. In the second part of the paper we shall treat of the errors to which an unsymmetrical setting such as is employed in these measurements is liable.

As usually described, in a form suitable for carrying out on an ordinary spectrometer without extra fittings, the critical angle method compares unfavourably with the minimum deviation method. It is somewhat less sensitive, the setting is not such a comfortable one in practice, and the satisfactory adjustment of the illumination requires much more care. In addition it is very wasteful of light and so is not suitable for use with faint lines. Without some compensating advantage the method would not find any place in connection with absolute determinations.

The minimum deviation method of refractometry involves the measurement of quite considerable angles to a high degree of accuracy. While many spectrometers profess an accuracy quite adequate for, say, fifth place certainty in refractometric determinations there is ample evidence in the published results of various workers that such accuracy is by no means always obtained. This is not due to lack of sensitivity or inability to repeat readings, which can be done with ample accuracy on comparatively small instruments, but to erratic behaviour of the centre and

* J. Guild, "Notes on the Pulfrich Refractometer," *Proc. Phys. Soc.* 30 (1918), 157.

various other defects. By following a proper routine in the determinations and taking sufficient precautions most of the systematic instrumental defects can be eliminated, but there is always the possibility of vagaries remaining to vitiate the result.

It is therefore desirable that results of importance should be obtained on more than one instrument and also by as many methods as possible so as to employ such instruments in different ways.

The following arrangement of the critical angle method has the very great advantage that it involves the measurement of small angles only, and so diminishes considerably the total quantity to be measured. This compensates sufficiently for its disadvantages in other respects to make it a useful alternative method for checking results otherwise obtained. It is unnecessary to describe here the principles underlying the critical angle method; these are adequately treated in various textbooks. We may say, however, in passing, that we are unaware of any circumstances in which it is not very much preferable to employ the boundary of zero refraction using a convergent beam at grazing incidence rather than the boundary of total reflection. For some reason the latter arrangement has pride of place in most practical textbooks.

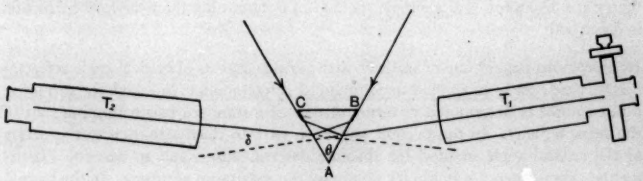

Fig. 1.

General Method.

In Fig. 1, let T_1 represent the telescope of the spectrometer. Although not essential it is an advantage to have it provided with a micrometer eyepiece. A second telescope T_2 is required. This can either be mounted on a separate stand with levelling screws or the collimator of the spectrometer may be utilised, the slit being replaced by an eyepiece with crosslines. The equilateral prism ABC is placed on the table of the spectrometer and levelled by the usual methods. The top of the prism should be approximately at the height of the axes of the telescopes. By an arrangement to be described presently a convergent beam of light is made to impinge tangentially along each of the faces BA and CA. Each beam is partially refracted into the prism and then refracted out again at the opposite face. The telescope T_1 is rotated until the band of light corresponding to the wave length for which the determination is to be made comes into the field, and is adjusted so that the crosslines are set on the edge which corresponds to rays incident on the face CA at grazing incidence. The second telescope, T_2, is then arranged so that it

receives the other emergent beam in the same manner. This procedure assumes that T_2 is on a separate stand: if the collimator is employed for T_2 and cannot be rotated, it is necessary to orient the prism in relation to T_2 so that the emergent beam passes approximately along its axis and then to adjust T_1 to the other beam. When all preliminary adjustments have been made the procedure is as follows: The critical edge in the field of the fixed telescope T_2 is brought to exact coincidence with the crosslines by a slight rotation of the spectrometer table, using the tangent screw which gives fine adjustment to the latter. The crosslines of T_1 are then set on the corresponding edge of the other beam. This should be repeated several times, *the adjustment of the edge in T_2 being made each time.*

A reading is thus obtained of the position of T_1 which corresponds to the direction of the critical ray emerging from BA when the critical ray emerging from CA is parallel to the optical axis of T_2. If T_1 is provided with a micrometer eyepiece it is best to clamp T_1 in approximate position at the commencement, subsequent settings being made with the micrometer. The scale reading must then be corrected for the mean micrometer reading.

The next process is to rotate T_1 into alignment with T_2. To do this the crosslines of T_2 must be illuminated by placing a suitable source of light (see later) behind the eyepiece. T_2 then acts as a collimator with respect to T_1 and an image of its crosslines is formed in the plane of the crosslines of T_1. The latter is adjusted until both sets of crosslines coincide, when the axes of the telescopes will be exactly in line. The angle through which T_1 has been moved, corrected for the difference in the micrometer readings, is the angle δ (Fig. 1) between the axes of the two telescopes when they are both correctly set on their respective critical edges. With ordinary glasses δ is always small so that only a little rotation of the telescope is required and large errors due to a defective centre are unlikely to arise.

If e is the angle of emergence which the critical rays make with the normals to the faces BA and CA and θ is the angle $B\hat{A}C$,

$$2e = \delta - \theta^{\bullet}.$$

If the process is repeated with each angle of the prism in turn, giving three nearly equal values δ_1, δ_2, δ_3, then

$$2e = \frac{\delta_1 + \delta_2 + \delta_3}{3} - 60°,$$

where e is the angle of emergence corresponding to a prism angle of 60° exactly†. Having determined e in this way, the refractive index is given by

$$\mu^2 = 1 + \left(\frac{\cos\theta - \sin e}{\sin\theta}\right)^2,$$

θ being the prism angle to which e corresponds.

• On the sign convention for prism formulae adopted by the authors (see Guild, *loc. cit.*) e is positive if the emergent ray is between the normal and the refracting edge of the prism. Under the circumstances of the present discussion e is always negative for glass prisms.

† In the absence of pyramidal error. If the prism has pyramidal error the sum of the three angles is $180° + \xi$, where $\xi = p^2 \cot 60°$, p being the inclination of an edge of the prism to the face opposite it. See Guild, "Note on Pyramidal Error in Prisms," *Trans. Opt. Soc.* 22 (1920–21), 139.

A slight variant of the method, which may be used if the spectrometer table is not provided with a convenient fine motion for making the settings with the telescope T_2, is to fit the latter also with a micrometer eyepiece. The prism is then left fixed, the settings of T_2 being made with its micrometer, which should finally be adjusted to the mean of these settings before T_2 is used as collimator for the collinear settings.

Practical Details.

Success with the critical angle method depends almost entirely on the illumination arrangements. If these are satisfactory the method is easy and accurate; but if they are otherwise, or if they require readjustment in the course of a determination, the greatest trouble may be encountered. If two similar sources are available the simplest scheme is to use one for each face, condensing lenses being employed to give convergent beams. A multiplicity of light sources is not always available, however, and the arrangement shown in Fig. 2 was designed to enable both faces to be simultaneously illuminated by one source. D is a hole about 3 mm. diameter, a little outside the focus of a condensing lens L. The source, if it is of appreciable width, is placed behind D, or an image of it is formed at D by another lens as if D were the slit of a spectroscope. The light leaves the lens L in a slightly convergent beam. It then meets a 60° prism, P, which should be of clear colourless glass polished on all three sides, but which need not otherwise be of good quality. This splits the beam, by internal reflection, into two beams which emerge in the directions shown. These beams, when they have separated to an extent of 15 cm. or so, meet a pair of mirrors or right-angled prisms so placed as to cause them to reconverge at an angle of 60°. The various parts can be mounted on a single stand, when they form a very convenient illuminator, easily adjustable in relation to the spectrometer. The supports of the right-angled prisms should be capable of rotation, as this facilitates adjustment. The illuminator is so placed in relation to the spectrometer that the prism under test occupies the position shown dotted in the figure. The lens L should have a power of about 10 or 11 dioptres, and the distance of D from it should be such that images of D are formed at D' and D'', about 5 cm. beyond the apex of the prism.

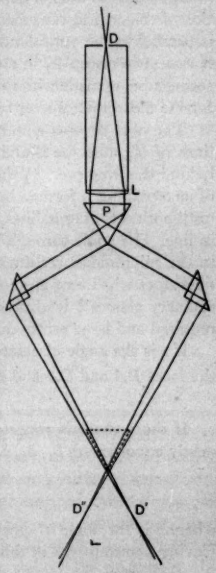

Fig. 2.

The simplest procedure for making the adjustments is as follows. The illuminator is set as nearly as possible in its correct position relative to the prism under test

(this may be facilitated by the use of a card cut to 60°) and the source of light placed behind the aperture *D*. The prism *P* should then be placed so that the two beams into which it splits the light are of equal cross-section and pass over the centres of rotation of the supports for the right-angled prisms. This is easily observed by intercepting the beams with a piece of ground glass. The reflecting prisms should be placed on their supports with their cathetal surfaces over the axes of rotation, and turned until they throw the converging beams along the faces of the test prism. If bright bands with sharp clean critical edges are not now obtained in the observing telescopes, a slight rotation of the reflectors while the corresponding bands are under observation should put them right. If a band is rather narrower than is desired, it can be broadened by moving the corresponding reflector slightly back. The height of the illuminator should be such that the light is incident on the prism faces as near the top as possible. This is necessary since the top of the prism is level with the middle of the aperture of the telescopes.

The foregoing description is given at some length because the success of the subsequent measurements depends on getting the adjustments correct in the first instance. If these are attempted by haphazard means endless difficulty is encountered; but if the routine suggested is followed the whole process is only a matter of minutes and success is certain.

The adjustments having been made, the settings on the critical edges of the bands will not usually present any difficulty. If the source is rich in light of wave lengths other than that for which the measurement is being made, stray light of these wave lengths will appear in the field and may give rise to chromatic parallax troubles. To overcome this difficulty coloured filters must be used in front of *D* to cut out or reduce the objectionable rays. The settings on the critical edges require the application of a small correction, which we shall discuss later.

In making the other measurement, in which one telescope acts as collimator with respect to the other, there are one or two precautions to be observed. In the first place it is not satisfactory to attempt to set the two crosslines in exact coincidence. Such a setting is very insensitive. The telescope axes should be very slightly out of line in the *vertical* direction so that one cross passes slightly above (or below) the other. Then when the axes are in line horizontally the two sets of wires form a small symmetrical rhomboid*. If a suitable vertical displacement is chosen, this rhomboidal setting can be made with great precision. In repeating the determinations *ab initio* the relative positions of the two sets of crosslines should be interchanged. By this means any error due to personal equation or to want of exact parallelism of the two sets of wires is eliminated and a true setting, corresponding to exact coincidence of the axes, is obtained.

The illumination of the crosslines of that telescope which serves as collimator may present some difficulty. When the telescopes are focussed on the edges of the bands they are, of course, collimated for the wave length in question. In the presence of the usual amounts of chromatic aberration they will not be in correct focus for other wave lengths. Unless the measurements are being made for lines near either

* It is assumed that the crosslines are mounted diagonally.

extreme of the spectrum it will generally be quite satisfactory for the collinear observations to put a table lamp behind the eyepiece of one of the telescopes with, possibly, a colour filter to approximate the colour to that for which the telescopes are focussed. If, however, the refracted light is not near the middle of the spectrum it is not usually possible to get sufficiently sharp focus in the collinear measurements by the use of filtered white light, and it becomes necessary to employ light of the wave length for which the index determination is being made. An arrangement which can always be employed with strong lines is to mount a small mirror close up to the eyepiece of one telescope and adjust it so that it sends the light which emerges from the eyepiece back into it to illuminate the crosslines. The adjustment has to be made with considerable nicety, so it is expedient to mount the mirror on the end of a lever pivoted on a suitable stand. It can then be swung out of position when it is desired to look into this eyepiece and swung back into position for the collinear measurement without upsetting its adjustment.

A simpler plan, if the source contains few lines so that by means of a suitable colour filter the one for which the measurement is being made can be rendered predominant, is to focus an image of the source on the ocular by means of a lens and mirror. It is not necessary to disturb the lens when once adjusted, but the mirror must be capable of easy removal and replacement. The advantage of this method over the previous one is that the adjustment of the mirror does not require to be nearly so precise, and, if a suitable lens is chosen, a fairly large circular area of the field is illuminated.

The necessity for devices of this description would, of course, be obviated, if makers would fit spectrometer telescopes with apochromatic or semiapochromatic objectives.

Special case of $\mu \fallingdotseq 1{\cdot}53$. When the refractive index of the prism is 1·528 approximately the angle δ is zero, and, for indices fairly near this value, δ is so small that it can conveniently be measured on the micrometer eyepiece without rotation of the telescope at all. In fact no spectrometer is necessary in this case: the telescopes may be mounted separately on rigid V stands and a suitable independent support, with the necessary adjustments for levelling, etc., may be used for the prism under test. It is, however, necessary that all such parts should rest on a rigid base. A wooden table is not good enough. A slate or glass slab, raised slightly from the table by three slips of rubber or cardboard, makes a useful base on which to erect such apparatus.

It is this property of the critical angle method, viz., that it can be arranged so as to require only a slight angular movement of the telescope, or in special cases no such movement at all, that invests the method with importance in precise refractometry as a check on other methods which, while more suitable for general work, involve the measurement of large angles. It is desirable therefore to investigate the errors to which the results may be liable on account of the special nature of the settings, and we shall discuss these in the following section.

Errors of Setting in the Critical Angle Method. The errors to which the type of setting peculiar to the critical angle method of refractometry are liable do not

appear to have been determined experimentally. The question has been debated and opinions expressed as to what the selected position of the crosswires does actually indicate; but beyond some experiments of fourth place accuracy made by Abbe, who compared the values of the indices of some liquids as obtained by the use of hollow prisms (minimum deviation method) with the results given on his well-known refractometer, not much of an experimental nature appears to have been done.

There are two points to be considered regarding the setting of crosswires on the edge of a bright band. In the first place such a setting is liable to errors on account of irradiation and personal equation, and, secondly, the band is not of uniform brightness right up to its edge.

We shall consider the latter point first. There is no actual discontinuity at the critical edge. The intensity of the refracted beam falls off more and more rapidly as the critical angle is approached and is actually zero at that angle. The intensity curve can be calculated from Fresnel's equations for refraction and reflection, correction being made for the fact that the rays in the refracted beam are compressed more and more closely as the critical angle is approached. These curves are exhaustively treated by Kruess*, and in Fig. 3 the intensity curve for a band 1° wide is given for a prism of refractive index 1·6. What will the eye adopt as the "edge" of such a band? There is reason for believing that the "edge" will be located where the intensity is diminishing most rapidly. While it is not evident from the figure, on account of the scale to which it is drawn, the slope of the curve increases right up to the critical angle and is therefore greatest at this point. Therefore, if the above supposition is correct, the setting should be the same as that made with a uniformly bright band ending abruptly at the critical angle.

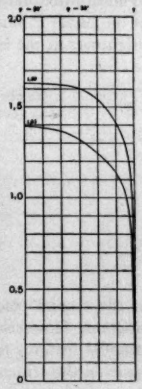

Fig. 3.

To test the validity of this conclusion we attempted to produce artificially conditions similar to those near the edge of the band. A circular rod, covered with white paper, was set up at about 20 feet from an observing telescope with a micrometer eyepiece. One side of this rod formed the object on which settings were made. In the first instance light was supplied by a "Pointolite" lamp at *P* placed close to the line of sight (Fig. 4). As observed in the telescope the rod appeared as a luminous band, the intensity of illumination falling off very rapidly towards the edge. This was due to the fact that when the light falls nearly at grazing incidence on the paper surface practically none of it is scattered directly backwards. While the distribution of intensity near the edge of this band is probably quite different from that in the case of the critical angle band the conditions are similar in two important particulars, viz., that the intensity is falling off very rapidly near the edge, being practically zero at the true edge, and also that the rate of diminution

* *Z. f. Instrk.* 39 (1919), 73.

of intensity is greatest at the true edge. Settings of the micrometer were made under these conditions and also when the rod was illuminated by a lamp at P'. In the latter case the band was very bright at the edge. Sometimes several lamps were arranged so as to make the illumination as nearly uniform as possible for some distance from the edge. A faintly illuminated ground glass was usually placed at G to provide a background, but on some occasions no background was used.

There was no detectable difference between the settings with the two methods of illumination. The telescope used had a power of 20 diameters, and the means of series of readings under the two conditions agreed to 0·2 second.

Unfortunately it is not possible to say that the arrangement with the lamp at P gave a band exactly analogous to the critical angle band. In the first place, in order to ensure that light actually reaches that part of the surface which forms the edge of the band, P must be brought a little to the side of the line of sight, and so the brightness at the edge of the band is not absolutely zero although it is very little.

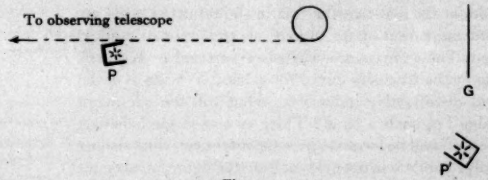

Fig. 4.

In the second place the distribution of brightness within the edge is probably different. However, the conditions are, in our opinion, sufficiently similar in their essential features for the result to be taken as indicating that the same setting is made on the critical edge as would be made if the intensity were uniform right up to that edge and then suddenly dropped.

It therefore only remains to determine the error due to irradiation and personal equation when setting on the boundary between a bright and a dark area. An arrangement which proves very convenient for this purpose is shown in Fig. 5. A prism P, which must have a sharp unchipped edge, is placed at the focus of a long focus telescope objective L with one of its faces approximately at 45° to the axis. A source of light S_1 and a sheet of ground glass G_1 are placed so as to illuminate the face of the prism which is turned towards L. A glass plate A is placed a little behind the prism as shown, and a second source of light S_2 and ground glass G_2 are placed so that A will appear illuminated when observed from L. The observing telescope is placed at T and focussed sharply on the edge E of the prism. If S_1 is on and S_2 is off the field is bright on one side of E and dark on the other, while if S_2 is on and S_1 off the bright and dark sides are reversed. If, then, the crosslines of the telescope are set on the edge when first one side and then the

other is bright, the difference in the micrometer readings will be twice the error of the setting.

We find that for each of us the error amounts to about 40 seconds divided by the magnification of the telescope, the setting being too far towards the dark side. Thus with a power of × 20 the error is about 2 seconds. *Further, this error is independent of brightness over a very wide range and also of colour.*

We obtained the same results when the dark side was faintly illuminated. In fact the error was unaffected until the two sides were nearly equal in brightness, when it rapidly diminished to zero.

By slightly modifying the arrangement of the apparatus we investigated the effect of varying the width of the band. The error remains practically constant until the angular width of the band is reduced below 4 minutes when it increases rapidly, becoming from three to four times as great for a width of 1 minute.

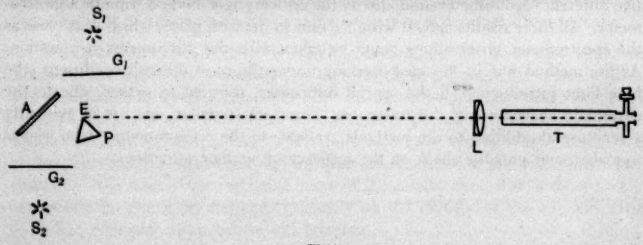

Fig. 5.

Thus the error of setting on an edge remains constant under all conditions liable to arise in practical observations, a fortunate and somewhat unexpected property which greatly facilitates the determination and application of the necessary correction. As regards the accuracy to which the truth of this statement may be regarded as established, many determinations of the error were made which varied among themselves from 1·3 to 2 seconds for a × 20 telescope, but these differences showed no systematic relation to the conditions of experiment and were within the experimental error of individual settings. We feel sure that our mean results would have indicated any systematic variations amounting to 10 or 15 per cent. of the total error.

These results are of importance in connection with commercial critical angle refractometers. In the discussion of the paper on the Pulfrich Refractometer already mentioned* the author expressed the opinion, based on indirect evidence, that the personal equation error with that instrument did not exceed a tenth of a minute. When properly eyepieced the power of the telescope of such a refractometer is at least × 8. The error in this case would be about 5 seconds. Thus the results here obtained bear out the conclusions previously formed from general considerations.

* *Proc. Phys. Soc.* 30 (1918), 157.

It has to be borne in mind, however, that if the edges of the bands are not in sharp focus in the plane of the crosswires—as may easily happen in fixed focus instruments—the error will be greater than that determined here and will no longer be independent of brightness.

Reverting to the general method, the observer should determine his own personal error by the method here given, using the telescope with which the determinations are to be made.

As an example of the utility of the arrangement described in the early part of this paper, and incidentally as furnishing a confirmation of our conclusions concerning the magnitude of the corrections to be applied to the settings, we may mention that we determined the index of a particular prism for the mercury line, $\lambda = \cdot 436\ \mu$, for which the value was in the neighbourhood of 1·528 (actually 1·52789 at 17° C. and 760 mm. pressure) by the minimum deviation method on two different spectrometers and also by the critical angle method without a spectrometer. All three results agreed within 1 unit in the fifth place, which is as close as the spectrometer observations could be taken with the instruments in question. As the method was in this case working under the most difficult conditions (the blue light introducing all the special difficulties, referred to earlier, due to the chromatic aberration of the telescopes, chromatic parallax, etc.) it is evidently a very useful addition to the methods available to the refractometrist who wishes to make every possible check on the accuracy of his determinations.

NOTE ON THE CORRECTIONS FOR TEMPERATURE AND ATMOSPHERIC PRESSURE IN REFRACTOMETRY

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(Optics Department, National Physical Laboratory.)

MS. received, 26*th November*, 1920. *Read*, 15*th December*, 1920.

Abstract. The paper gives the relationship between the relative refractive index of a substance in air at a given temperature and pressure and the absolute index of the substance in forms convenient for the correction and correlation of results.

As is well known the refractive constants of transparent materials vary with temperature, and in results professing any high degree of accuracy the temperature of the observation is usually stated. It is by no means so generally realised that the results as obtained by practically all the methods employed for precise work depend also on the atmospheric pressure and require correction for the height of the barometer. This arises from the fact that the quantity measured in practice is the *relative index*, i.e., the ratio of the refractive index of the substance to that of the air which surrounds it. The latter depends primarily on the density of the air, and varies therefore both with temperature and pressure.

Although this fact is bound to be well known it is quite evidently overlooked by most workers on refractometry. The only published references to the effect of barometric height which the writer has seen are in one of Gifford's papers in which he mentions that its significance has been pointed out to him and makes retrospective calculations of the corrections required for some of his determinations, and a reference by the writer of this note himself in reply to the discussion on a paper on the Pulfrich Refractometer*. I know of no published tables of refractive indices in which is stated the pressure as well as the temperature for which the results hold. It may be useful therefore to call specific attention to this point, and at the same time mention the method of connecting the various quantities involved which has been found most convenient at the Laboratory.

The variation of absolute index $\bar{\mu}$ with temperature may be taken as linear over the usual range of laboratory temperatures, so that

$$\bar{\mu}_{t'} = \bar{\mu}_t \{1 + \alpha (t' - t)\},$$

where α is the temperature coefficient of absolute index and is taken as positive when the index increases with temperature. Let us represent the refractive index of the air by n. The variation of n with temperature and pressure follows Gladstone and Dale's law that $n - 1$ is proportional to the *density*.

* *Proc. Phys. Soc.* 30 (1918), 157.

Thus $$n_t^p - 1 = (n_0^{760} - 1)\frac{p}{760}\cdot\frac{273}{273+t} = \beta\left(1+\frac{\delta}{760}\right)\left(1-\frac{t}{273}\right)$$

$$= \beta + \beta\frac{\delta}{760} - \beta\frac{t}{273},$$

where $\beta + 1$ is the refractive index of the air at 0° C. and 760 mm. pressure and δ mm. is the *excess* of pressure above 760 mm.

Let μ_t^{δ} be the observed relative index at $t°$ and $760 + \delta$ mm. pressure, then

$$\bar{\mu}_t = \mu_t^{\delta} n_t^{\delta} = \mu_t^{\delta}(1 + n_t^{\delta} - 1)$$

$$= \mu_t^{\delta} + \mu\beta + \mu\beta\frac{\delta}{760} - \mu\beta\frac{t}{273} \qquad \text{.....(i).}$$

This gives $\bar{\mu}_t$ in terms of the observed relative index.

Similarly for any other temperature and pressure

$$\bar{\mu}_{t'} = \mu_{t'}^{\delta'} + \mu\beta + \mu\beta\frac{\delta'}{760} - \mu\beta\frac{t'}{273} \qquad \text{.....(ii).}$$

Subtracting (ii) from (i) and rearranging, we get

$$\mu_{t'}^{\delta'} = \mu_t^{\delta} + \mu\beta\frac{\delta-\delta'}{760} - \mu(t-t')\left(\alpha + \frac{\beta}{273}\right) \qquad \text{.....(iii).}$$

Equation (iii) gives the relative index at any temperature and pressure in terms of the relative index at any other temperature and pressure by the addition of separate small corrections for the differences in temperature and pressure. It is evident that the temperature coefficient of relative index is $\alpha + \beta/273$.

These relationships enable us to express a result either as absolute index at a standard temperature or as relative index at a standard temperature and pressure, provided we know the temperature coefficient. The most convenient form for the result is relative index at, say, 17° C. and 760 mm. pressure. If the temperature coefficient is not known, as is usually the case at present, the results should be expressed as relative index at the temperature of the observation and at 760 mm. pressure.

The values of β can be obtained from the results of Kayser and Runge*, Scheel†, and others. Kayser and Runge's formula for *damp* air is

$$\beta = 10^{-7}\left(2878{\cdot}7 + 13{\cdot}16\frac{1}{\lambda^2} + 0{\cdot}316\frac{1}{\lambda^4}\right),$$

where λ is the wave length of the light. For dry air 3×10^{-7} has to be added. This difference is quite negligible for our present purpose. The following table gives the values for dry air for various spectrum lines.

* *Abhandl. d. Berl. Akad. Wiss.* 1893.
† *Berl. Verh. d. D. Phys. Ges.* 9 (1907).

Designation in Solar Spectrum	Approximate λ	β
A	7590 A.U.	·0002905
B	6870 „	11
C	6560 „	14
D	5890 „	22
E	5270 „	33
F	4860 „	43
G	4310 „	62
H	3970 „	78

From equation (iii) we see that for each millimetre change in barometric pressure the relative index varies by $\mu\beta/760$ or, taking μ as 1·5, by 6×10^{-7}. Thus a change of pressure of about 17 mm. will produce a change of 1 unit in the fifth decimal place, which is the last significant figure in precise determinations at the present time.

Since the temperature coefficients of optical glasses rarely exceed from 1 to 2 units in the fifth place per degree Centigrade, it is clear that for results of fourth place accuracy it is unnecessary to specify barometric pressure and scarcely necessary to specify temperature; but for work claiming fifth place accuracy or better it is clearly essential to specify both.

A SIMPLE DIFFERENTIAL REFRACTOMETER FOR LIQUIDS

By JOHN S. ANDERSON, F.Inst.P.
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Read, 15th December, 1920.

Abstract. A number of sensitive differential methods for liquids are briefly discussed and a simple new method, which is applicable to a number of problems, is described.

For the determination of the refractive indices of a liquid a number of methods, giving various degrees of accuracy, are available. The liquid may, for example, be placed in a hollow prismatic cell and the angles of minimum deviation for the standard wave lengths measured by means of a spectrometer. The Abbe and Pulfrich refractometers, which depend on critical angle methods, can also be utilised for the measurement of the indices of liquids; in the same class one may include the dipping refractometer*, which is specially designed for liquids.

It frequently happens, however, that it is more important to be able to determine the differences between the indices of two similar liquids than to find the absolute indices of either. The problem arises, for example, in the case of the relative salinity of different samples of sea water† and in the case of the relative purity of different samples of glycerine. The differences met with in such cases are very small, so that it is necessary to use a method which will yield more accurate results than can be obtained with the ordinary direct methods mentioned above.

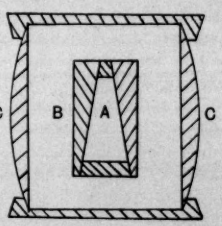

Fig. 1.

Before discussing differential methods it may be of interest to refer to Féry's method of measuring the refractive index of a liquid, as it is not usually described in textbooks. Féry‡ employed a double cell such as is shown in Fig. 1. The liquid under examination was placed in the inner cell *A* and the space *B* between the inner and outer cells was filled with water, which served to keep the liquid at uniform temperature. The two walls *C*, *C* of the outer cell consisted of low power plano-convex lenses. The compound cell was mounted between a collimator and telescope and was moved

* C. Pulfrich, *Z. f. Instrk.* 19 (1899), 335.
† This problem was suggested to the author by Mr D. J. Matthews of the Admiralty Scientific Research and Experiment Department.
‡ C. Féry, *C. R.* 113 (1891), 1028; 119 (1894), 332: see also C. Chéneveau, *Ann. de Chim. et Phys.* 12 (1907), 175 for descriptions of this and other methods applicable to liquids.

at right angles to their common axis until the deviation of a narrow pencil of light by the glass walls of the inner cell and by the liquid was compensated by the deviation due to the plano-convex lenses. The difference between the indices of the liquid and of the glass of the inner cell walls could then be determined from the amount of displacement of the double cell. This displacement could be determined to 1/300 mm., which corresponded approximately to a difference in refractive index of one in the fourth decimal place. The method thus consists essentially in compensating the deviation due to a liquid prism of constant angle by the deviation due to a glass prism of variable angle formed by the plano-convex elements.

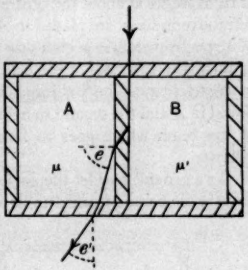

Fig. 2.

An accurate differential method has been described by Hallwachs*. The form of cell employed in this method is illustrated in Fig. 2. It consists of a rectangular cell divided into two compartments A, B by a plane parallel piece of glass set at right angles to two of the cell walls. The two liquids under examination are placed in the compartments A, B and a beam of light is allowed to enter at grazing incidence at the surface of separation between the partition and the liquid of lower refractive index. Let μ, μ' be the indices of the liquids in compartments A, B respectively, and e, e' the angles of emergence of the critical rays in the liquid of higher refractive index (μ) and in the air. Then

$$\mu \sin e = \mu' \quad \text{and} \quad \sin e' = \mu \cos e.$$

Thus
$$\sin^2 e' = \mu^2 - \mu'^2,$$

or
$$\mu - \mu' = \frac{\sin^2 e'}{\mu + \mu'}.$$

A difference in index of one in the fifth decimal place corresponds, therefore, in the case of liquids having indices near that of water, to an angle of emergence of about 1/3 of a degree. If the partition wall is not perfectly plane parallel any prismatic effect thus introduced may be eliminated by turning the cell through 180°, measuring the double deviation, and taking half this angle as the value of e'.

Another very sensitive differential method is that of the interference refractometer, such as was employed by the late Lord Rayleigh† for measuring the refractive indices of gases. The principle of the method is to determine the optical path difference of equal layers of two liquids by measuring the relative displacement of a system of interference fringes formed with white light. A difference in refractive index of one or two units in the sixth decimal place may be detected by this method.

For many purposes the above methods are much more sensitive than is required,

* W. Hallwachs, *Ann. d. Phys. u. Chem* 50 (1893), 577.
† Lord Rayleigh, *Phil. Mag.* 33 (1917), 161.

and, further, they offer a number of difficulties in their manipulation. The author has devised a simple differential method which, although not so sensitive as the others, is sufficiently accurate for a large-class of measurements. It has already been applied with success in one or two cases. The type of cell required is illustrated in Fig. 3. The two walls which divide the cell into two compartments A, B meet at right angles at about the centre of the cell. If two liquids, having slightly different refractive indices, are placed in A and B and the cell is mounted on the prism table of a spectrometer, in such a way that the parallel beam of light from the collimator is normal to the sides of the cell, two images of the collimator slit will be observed in the telescope. The first image will be due to light which passes direct through the liquid in A and the second to light which passes through both liquids. The deviation of the beam which goes to form the second image may easily be calculated as follows.

As a general case let the angle between the inner walls of the cell be 2θ (Fig. 4), these walls being assumed to lie symmetrically with respect to the walls of the cell.

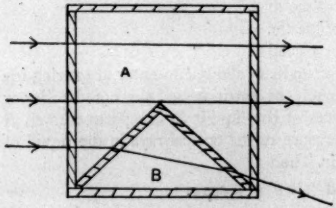

Fig. 3.

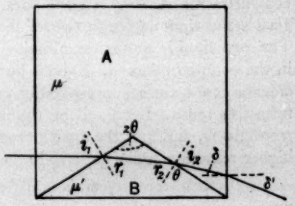

Fig. 4.

Let μ, μ' be the refractive indices of the two liquids in A, B respectively, i_1, i_2 the angles in the medium μ which a ray, initially normal to one of the sides of the cell, makes with the normals to the inner walls, r_1, r_2 the angles in the medium μ' which the ray makes with the same normals, and δ, δ' the final deviations produced in the medium μ and in air respectively. Then

$$\mu \sin i_1 = \mu' \sin r_1 \qquad \mu' \sin r_2 = \mu \sin i_2 \qquad \mu \sin \delta = \sin \delta'.$$

Also
$$r_1 + r_2 = 2\theta \qquad i_2 = \theta + \delta \qquad i_1 = \theta.$$

Hence

$$\sin \delta' = \mu \sin (i_2 - \theta) = \mu (\sin i_2 \cos \theta - \cos i_2 \sin \theta)$$
$$= \mu' \sin r_2 \cos \theta - \sqrt{\mu^2 - \mu'^2 \sin^2 r_2} \sin \theta.$$

Now

$$\mu' \sin r_2 = \mu' \sin (2\theta - r_1) = \mu' (\sin 2\theta \cos r_1 - \cos 2\theta \sin r_1)$$
$$= \sqrt{\mu'^2 - \mu^2 \sin^2 \theta} \sin 2\theta - \mu \sin \theta \cos 2\theta.$$

Thus sin δ' may be expressed as a function of μ, μ', and θ. In the particular case for which $2\theta = 90°$ we have

$$\mu' \sin r_2 = \frac{1}{\sqrt{2}}\sqrt{2\mu'^2 - \mu^2}$$

and

$$\sin \delta' = \tfrac{1}{2}\sqrt{2\mu'^2 - \mu^2} - \tfrac{1}{2}\sqrt{3\mu^2 - 2\mu'^2}.$$

If we put $\mu' - \mu = d\mu$ this reduces to

$$\sin \delta' = 2d\mu \left\{1 + \frac{1}{2}\frac{d\mu}{\mu} + 2\left(\frac{d\mu}{\mu}\right)^2 + 3\left(\frac{d\mu}{\mu}\right)^3 + \ldots\right\} \qquad \ldots\ldots(1).$$

Thus, if the difference in refractive indices is very small we obtain the very simple formula

$$\delta' = 2d\mu \qquad \ldots\ldots(2),$$

since the angle of deviation δ' is very small. As an example, the angle of deviation corresponding to a difference of one in the fifth decimal place of refractive index is about 4 seconds. If one is dealing with comparatively large differences of refractive index, formula (1) will yield accurate results provided the necessary number of terms of the infinite series is employed.

When dealing with small differences of index the method has the following advantages.

(1) A knowledge of the refractive index of either of the two liquids is not required.

(2) The deviations are so small that they may very conveniently be measured by means of a micrometer eyepiece in the observing telescope.

(3) The method is simple and easy of manipulation.

(4) The calculations involved in determining differences in refractive index are simple.

For small deviations the sensitivity may be doubled by using an autocollimating telescope and placing a plane silvered mirror at the other side of the cell in such a position that the axis of the telescope is normal to the reflecting surface. In this case a difference in refractive index of approximately one in the fifth decimal place could be measured.

The reasons for using an angle of 90° between the inner walls of the cell, rather than a larger angle which would give a bigger deviation, are that the right angle cell is easier to construct and the formula connecting δ' and $d\mu$ is simpler. It may be found more convenient to use two square glass cells, one being placed inside the other in such a position that its walls make angles of 45° with the walls of the outer cell.

As a practical detail it is better to use a fine line parallel to the length of the slit of the collimator or autocollimating telescope than to use a narrow slit. The slit can then be quite wide and what will be observed will be two images of the line on a bright field. With such an arrangement one can make more accurate determinations of the deviation.

If the index difference is small it is possible to make the necessary measurements

with white light, as the dispersion introduced will be very small. For more accurate determinations, however, the slit of the collimator or autocollimating telescope should be illuminated with suitable monochromatic light.

In order to avoid temperature differences in the two liquids the cell should be covered and suitably protected from draughts. The necessity for very accurate temperature control may be realised from the fact that in the case of distilled water at room temperatures a difference in index amounting to one in the fifth decimal place corresponds approximately to a difference in temperature of $1/7°$ C. Thus the problem of measuring minute differences in the refractive indices of liquids, say one in the sixth decimal place, is bound up with the problem of maintaining the liquids at the same temperature to within very small limits. It is evident, therefore, that in order to measure small differences of index in the case of liquids some form of differential method is necessary, so that the liquids may be brought into close proximity with one another, and further, there is no point in using any method which is more sensitive than the one just described unless great precautions are taken to keep the liquids at the same temperature.

NOTES ON PHOTOGRAPHIC SHUTTER TESTING

By JOHN S. ANDERSON, F.Inst.P.
(Optics Department, National Physical Laboratory)

Read, 15th December, 1920.

Abstract. The paper briefly describes a number of the more important methods of measuring shutter speeds and in particular the method employed at the National Physical Laboratory. Details of a number of recent modifications are given and a method for determining data connected with the motion of the blind of a focal plane shutter is described.

The problem of accurately measuring the speeds of camera shutters has engaged the attention of a great number of investigators, but the methods evolved, though differing greatly as regards the nature and complexity of apparatus employed, are all based on the measurement of small intervals of time.

The term "speed" really denotes the total duration of exposure at any point of the plate. In the case of between-lens sector shutters this is equivalent to the total time during which the shutter leaves are open. In the case of focal plane shutters the speed is usually specified as the time taken for the opening of the blind to pass a point in its own plane opposite the centre of the plate. The duration of exposure at any other point usually has a somewhat different value owing to the variable velocity with which the blind moves. These durations in the plane of the blind differ from the intervals for which the corresponding points of the plate itself are exposed.

The speeds of ordinary shutters vary from about 1 to 1/1000 second, though some types of shutters are designed to give speeds of the order of 1 to 5 seconds. In order, therefore, that a method should give accurate results over the ordinary range of speeds it should be capable of measuring intervals of time to an accuracy of from 1/1000 to 1/10000 second. It should be noted, however, that for most purposes it is only necessary to obtain results accurate to about 1 per cent., since in most shutters the speed corresponding to any given setting varies perceptibly in successive exposures.

Useful information on simple methods of shutter testing, such as may be employed by amateurs who do not have complicated apparatus at their disposal, is to be found in recent volumes of the *British Journal of Photography*, in the German work, L. David's *Praktikum*, 3rd edition, and in a paper by P. Schrott in the *Photographische Korrespondenz*, Oct. 1919.

For more accurate determinations of shutter speeds a number of devices, such as rotating discs provided with spokes, vibrating flames, and tuning forks, have been employed for giving the necessary short intervals of time. The first named

arrangement was used by Abney*, the spokes which project from a rotating disc being made to interrupt a beam of light which passes through the shutter opening. The speed of rotation of the disc was determined by means of the note given out by a siren.

Methods of measuring shutter speeds by utilising the motion of a revolving disc, containing a slit or slits, across a stationary illuminated slit, have been described by Kershaw† and Hunter‡.

Benoist§ has utilised the vibrations of an acetylene gas flame to give equally spaced short intervals of time. A box is provided on one face with a small burner, at the extremity of which the acetylene led into the box is lighted. The opposite face, which is formed by a rubber membrane, is fixed against an opening arranged at a convenient height in the wall of an organ pipe of known frequency. When the organ pipe vibrates under the action of suitable bellows, the flame vibrates with the same frequency. A somewhat simpler arrangement is to use a flame of acetylene emitted from the end of a pointed tube which is surrounded by a glass chimney. Such a flame "sings" and the note can be varied by altering the length of the chimney, the frequency of vibration of the flame being determined by means of the note. Benoist‖ also suggests the employment of the vibrations of an electric arc run on alternating current of known frequency.

A method employed by the Eastman Kodak Company¶ consists in taking a number of photographs of the shutter during its period of operation, the time of exposure of each photograph being small in comparison with the rate of movement of the shutter. A beam of light from an arc lamp in a small projection lantern is reflected from a lightly mounted aluminium crown on which are placed 20 small plane mirrors held vertically on the face of the crown; the lantern condenser focusses an image of the arc crater at the mirror surface. The crown of mirrors is rotated about a vertical axis at a speed of 50 revolutions per sec. by means of a motor controlled by a centrifugal governor. The beam of light is thus interrupted 1000 times per sec., a frequency which has been found most suitable for general testing. The beam is limited at the condenser by a vertical slit 2 mm. wide, the width of the beam as it flashes by reflection past the shutter opening being about 1/30 of the distance between the flashes; thus each exposure is about 1/30000 sec. The reflected beam falls on a simple lens behind which the shutter to be tested is held, and a small camera lens of 90 mm. focal length forms an image of the shutter on the rim of an aluminium wheel 12 inches in diameter, around which a band of sensitive cinematograph film is fastened. This wheel is then rotated about a vertical axis by means of a crank and gearing at such a speed as to separate the different images of the shutter. Both the speeds and the efficiencies of a shutter may be determined by this method. For recording shutter speeds of 1/10, 1/5, and 1/2 sec. the image of the shutter opening is restricted to a narrow band by inserting a 1 mm. slit in

* Sir W. Abney, *Treatise on Photography* (1905), 290.
† A Kershaw, Patent No. 16053 (1904).
‡ J. de G. Hunter, *Trans. Opt. Soc.* 8 (1906), 1.
§ L. Benoist, *Brit. Journ. of Phot.* 58 (1911), 949.
‖ L. Benoist, *Bull. Soc. Fr. Phot.* (1910), 390.
¶ P. G. Nutting, *Brit. Journ. of Phot.* 63 (1916), 394.

front of the box in which the moving film is enclosed. In order to save counting the hundreds of images obtained at these speeds one of the twenty mirrors may be painted black.

A simple instrument for testing shutters has recently been put on the market by a German firm, Carl Janser, Junr. of Stuttgart, under the name "Columbus*." It consists of a tuning-fork, giving 50 complete vibrations per sec., between the prongs of which there is a small electric bulb. One of the tuning-fork prongs is L-shaped and a microscope objective, attached to the short limb, forms a sharp image of the lamp filament. The instrument is set up in front of the camera, whose shutter is to be tested, so that, when the fork is at rest, the image of the filament is focussed on the ground glass. A horizontal slit is inserted in front of the camera objective; when the fork is caused to vibrate, the illuminated point image is drawn out into a horizontal line. If, now, an exposure is made while the dark slide is moved upwards in the camera, a wavy line is obtained on the photographic plate, the speed of the shutter being determined from the number of waves formed.

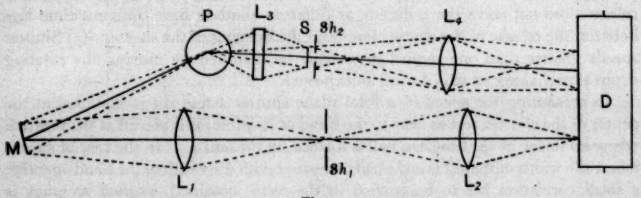

Fig. 1.

A very accurate method of testing shutter speeds, depending on the use of electrically maintained tuning-forks and vibration galvanometers, has been described by Campbell and Smith†.

The original method has been modified in a number of details in recent years: it may be of interest, therefore, to describe briefly the arrangements that are at present employed at the National Physical Laboratory. A plan of the optical system is shown diagrammatically in Fig. 1. The light from a "Pointolite" tungsten arc P is reflected from the mirror M of a vibration galvanometer and then passes through two lenses L_1 and L_2. A diminished point image of the incandescent tungsten ball is thus formed on the surface of a drum D which can be rotated about a horizontal axis at any convenient speed by means of a small-power motor with variable gearing. The shutter, whose speed is to be measured, is placed at the position Sh_1, where an image of the tungsten ball is formed by the first lens L_1, and a piece of sensitive bromide paper is wound round the drum. When the vibration galvanometer, which is in tune with an electrically driven tuning-fork of known frequency, is set in

* *Z. f. Feinmechanik*, 28 (1920), 58.

† A. Campbell and T. Smith, *Proc. Phys. Soc.* 22 (1907–9), 788; *N.P.L. Collected Researches*, 6 (1910), 93; see also T. Smith, *Trans. Opt. Soc.* 12 (1910–11), 128.

action, the mirror M vibrates about a vertical axis, so that the image on the surface of the drum is caused to vibrate in a horizontal plane. A wavy image is thus imprinted on the bromide paper when the drum is rotated, the number of waves depending on the total duration of exposure of the shutter and the frequency of the tuning-fork employed.

In the original form of the experiment a Nernst lamp was used as the source of light and in order to obtain a point image a fine horizontal slit was placed in front of a falling plate mechanism which was used for recording purposes. The present arrangement has the following distinct advantages over the original one. (1) The actinic value of the "Pointolite" lamp is much greater than that of the Nernst lamp, so that it is possible to use photographic paper instead of plates for making the records. It is much more convenient to measure records on paper than on plates, especially if the records are at all faint. (2) The use of a rotating drum obviates the possibility of making blank exposures: with the falling plate system it is sometimes difficult to know when a shutter should be released, with the result that portions of the time traces may not fall on the plate. The provision of an automatic release does not solve the problem, as different shutters have different time lags between the release of the shutter lever and the opening of the shutter. (3) Shutter speeds greater than one second may easily be recorded by moving the rotating drum slowly along its axis during an exposure.

In measuring the speed of a focal plane shutter a fine slit is mounted at the centre of the shutter and as near to the blind as possible: it is placed at the position where an image of the tungsten ball is formed by the lens L_1. In the case of speeds where the width of the slit is not small compared with the width of the blind opening, a small correction has to be applied to the result obtained, if great accuracy is required.

In order to measure the efficiency of a sector shutter a time trace is formed as in the case of speed measurement, but the shutter is placed at Sh_2 instead of Sh_1. In addition a horizontal slit S is placed in a diametral plane of the shutter and as close to the shutter leaves as possible. This slit is illuminated by light from the same "Pointolite" lamp P as is used for the time trace, a cylindrical lens L_3 being employed to give a drawn-out image of the tungsten ball. An image of the slit is formed on the surface of the drum by means of a lens L_4: it is advisable to arrange that the images of the slit and the tungsten ball should lie on the same horizontal line. The time trace may be limited to one complete revolution of the drum by inserting an auxiliary shutter, set at the necessary speed, between the lenses L_1 and L_2; the two shutters can then be released at practically the same instant*.

For testing speeds over the whole range commonly met with two tuning-forks giving 50 and 250 vibrations per second and a bar giving 1000 vibrations per second have been used at the National Physical Laboratory. Each of these is electrically maintained by means of microphone hummers† and is in tune with a vibration

* For the method of deducing the efficiencies from the traces obtained see A. Campbell and T. Smith, *loc. cit.*

† See A. Campbell, *Proc. Roy. Soc.* A 78 (1906), 208; *N.P.L. Collected Researches*, 2, 268.

galvanometer of corresponding frequency. The use of microphone hummers, however, gives a good deal of trouble in practice and is being superseded by a method which has been made possible by the development of triode valves in connection with wireless telegraphy*.

The method is illustrated diagrammatically in Fig. 2. The filament *F* of the triode valve *V* is heated by the battery *FB*. The anode *A* is connected through the anode battery *AB* and the coil *AC* to one terminal of the filament, and the grid *G*

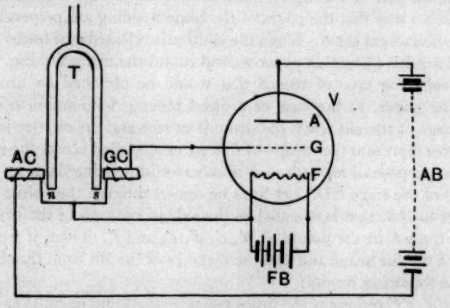

Fig. 2.

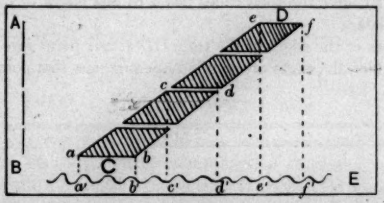

Fig. 3.

is connected to the filament through the coil *GC*. The tuning-fork *T* is permanently magnetised by an auxiliary magnet, on whose pole pieces the coils *AC* and *GC* are wound, the poles of the fork being indicated by *n* and *s*. Suppose the fork to have been set in vibration, and consider the moment when the two prongs are moving away from their respective coils. The motion of the pole *s* will induce an E.M.F. in the coil *GC* which will raise the potential of the grid and so increase the current flowing into the valve at the anode. This current leaving by the filament completes its circuit through the coil *AC*, round which it passes in such a direction as to repel

* W. H. Eccles, *Proc. Phys. Soc.* 31 (1919), 269; W. H. Eccles and F. W. Jordan, *Electrician*, 82 (1919), 704; for theory, see S. Butterworth, *Proc. Phys. Soc.* 32 (1920), 345.

the pole *n*, and thus assist its motion. The energy necessary for keeping the vibration galvanometer in action may be tapped off from a coil placed near *AC*. So far this method has been applied with success to the case of a tuning-fork giving 1000 vibrations per second and it is hoped that soon it may also be installed for forks with lower frequencies.

The method of measuring the efficiencies of sector shutters has been applied by the author to the case of focal plane shutters in order to obtain information with regard to the motion of the blind. The shutter to be tested is mounted at Sh_2 (Fig. 1) in such a way that the edges of the blind opening are perpendicular to the length of the horizontal slit *S*. When the shutter is released two traces are obtained on a piece of sensitive bromide paper wound round the drum *D*. Fig. 3 represents diagrammatically the type of record that would be obtained on unwinding and developing the paper, in the case of a blind moving with uniform acceleration. *AB* is the image of the slit when the drum is at rest and the shutter is fully open; the clear spaces represent the images of fine wires stretched across the slit at certain points for the purpose of reference. A number of interesting data can be obtained from a study of the trace *CD*. Let lines be drawn through the points *a*, *b*, *c*, *d*, *e*, and *f* parallel to *AB*, that is, parallel to the axis of rotation of the drum, so as to cut the time trace *E* in the points *a'*, *b'*, *c'*, *d'*, *e'*, and *f'*. Then, if we neglect the finite width of the slit image and also the distance of the slit from the shutter blind, we obtain the following results:

(1) *a'b'*, *c'd'*, *e'f'* represent the times taken by the blind opening to pass certain points such as the top, middle, and bottom of the plate.

(2) *a'e'* and *b'f'* give the total times taken by the edges of the blind opening to traverse the plate

(3) The slopes of the edges of the trace *CD* at any point give measures of the velocities with which the edges of the blind opening pass that point.